ON THE DIOPHANTINE EQUATION $N_n = x^a \pm x^b + 1$

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ABSTRACT. In this note we solve the Diophantine equation $N_n = x^a \pm x^b + 1$, where N_n denotes the n-th Narayana number, a, b are nonnegative integers with $0 \leq b < a$ and $2 \leq x \leq 30$.

1. INTRODUCTION

Narayana's cows sequence $\{N_n\}_{n\geq 0}$ is a ternary recurrent sequence given by the recurrence relation

$$N_{n+3} = N_{n+2} + N_n,$$

with seeds $N_0 = 0, N_1 = 1, N_2 = 1$. It is named after a 14th- century Indian mathematician Narayana Pandit [1] and the sequence first appeared in the book "*Ganita kaumudi*". The OEIS (Online Encyclopedia of Integer Sequences) number of $\{N_n\}_{n\geq 0}$ is A000930. The first few terms are

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \cdots.$$

The characteristic equation of $\{N_n\}_{n\geq 0}$ is $f(x) = x^3 - x^2 - 1 = 0$ and the characteristic roots are:

$$\alpha = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

$$\beta = \frac{1}{3} + w\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

$$\gamma = \bar{\beta} = \frac{1}{3} + w\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

where $w = \frac{-1+i\sqrt{3}}{2}$. The closed form known as the Binet's formula is given by

$$N_n = X\alpha^n + Y\beta^n + Z\gamma^n \text{ for all } n \geq 0,$$

with

$$X = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)},\ Y = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)},\ Z = \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}.$$

This can also be rewritten as $N_n = C_\alpha\alpha^{n+2} + C_\beta\beta^{n+2} + C_\gamma\gamma^{n+2}$ for all $n \geq 0$ where $C_x = \frac{1}{x^3+2}$ for $x \in \{\alpha, \beta, \gamma\}$. The minimal polynomial of C_α is $31x^3 - 31x^2 + 10x - 1$ and all the zeros of this polynomial are inside the unit circle. One can approximate the following:

$$\alpha \approx 1.46557;\ |\beta| = |\gamma| \approx 0.826031;\ |C_\beta\beta^{n+2} + C_\gamma\gamma^{n+2}| < 1/2 \text{ for all } n \geq 1.$$

Using induction it is easy to prove that

$$\alpha^{n-2} \leq N_n \leq \alpha^{n-1} \text{ holds for all } n \geq 1. \tag{1.1}$$

Recently, many research work have been done involving the term of a linear recurrent sequence and sum or difference of powers of two or three primes. For example, Marques and Togbé find all Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c$, in nonnegative integers a, b, c, with $\max\{a, b\} < c$. Luca and Szalay [5] proved that the Diophantine equation $F_n = p^a \pm p^b + 1$ admits many effectively computable positive integer solutions (n, p, a, b) where p is a prime number, $n > 2$ and $\max\{a, b\} \geq 2$. In [8], Rihane et al. studied all Padovan and Perrin numbers of the form $x^a \pm x^b + 1$.

In this work, we are interested to find Narayana numbers of the form $x^a \pm x^b + 1$. In particular, we solve the exponential Diophantine equation

$$N_n = x^a \pm x^b + 1. \tag{1.2}$$

Our main theorem is the following.

Theorem 1.1. *All the solutions of* (1.2) *satisfy* $a < n < 1.8 \cdot 10^{32} (\log x)^4$. *Furthermore the only solutions of* (1.2) *in positive integers* (n, x, a, b) *with* $0 \leq b < a$ *and* $2 \leq x \leq 30$ *are given by*

$$(n, x, a, b) \in \left\{ \begin{array}{l} (6,2,1,0), (7,2,2,0), (7,4,1,0), (8,7,1,0), \\ (9,2,3,2), (9,3,2,1), (9,11,1,0), (10,2,4,1), \\ (10,17,1,0), (11,26,1,0), (12,2,5,3), (22,12,3,2), \\ (23,7,4,3) \end{array} \right\}$$

for $x^a + x^b + 1$ *and*

$$(n, x, a, b) \in \left\{ \begin{array}{l} (4,2,1,0), (5,2,2,1), (5,3,1,0), (6,2,2,0), \\ (6,4,1,0), (7,6,1,0), (8,2,4,3), (8,3,2,0), \\ (8,9,1,0), (9,2,4,2), (9,4,2,1), (9,13,1,0) \\ (10,3,3,2), (10,19,1,0), (11,28,1,0), (15,2,8,7) \end{array} \right\}$$

for $x^a - x^b + 1$.

2. Preliminaries

Baker's concept performs a vital role in reducing the bounds concerning linear forms in logarithms of algebraic numbers. Let η be an algebraic number with minimal primitive polynomial

$$f(X) = a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient $a_0 > 0$, and $\eta^{(i)}$'s are conjugates of η. Then, the *logarithmic height* of η is given by

$$h(\eta) = \frac{1}{d} \left(\log a_0 + \sum_{j=1}^{d} \max\{0, \log |\eta^{(j)}|\} \right).$$

The following are some properties of the logarithmic height function which will be used later in our proof.

$$h(\eta+\gamma) \leq h(\eta)+h(\gamma)+\log 2,$$
$$h(\eta\gamma^{\pm 1}) \leq h(\eta)+h(\gamma),$$
$$h(\eta^k) = |k|h(\eta),\ k \in \mathbb{Z}.$$

The following theorem of Matveev (see [7] or [2, Theorem 9.4]) provides a large upper bound for the subscript n in (1.2).

Theorem 2.1. *Let $\eta_1, \eta_2, \ldots, \eta_l$ be positive real algebraic integers in a real algebraic number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$ and $b_1, b_2, \ldots, b_l$ be non zero integers. If $\Gamma = \prod_{i=1}^{l} \eta_i^{b_i} - 1$ is not zero, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1+\log d_{\mathbb{L}})(1+\log D) A_1 A_2 \ldots A_l,$$

where $D = max\{|b_1|, |b_2|, \ldots, |b_l|\}$ and $A_1, A_2, \ldots, A_l$ are positive real numbers such that

$$A_j \geq max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \text{ for } j = 1, \ldots, l.$$

The following result of Baker and Davenport due to Dujella and Pethő [3, Lemma 5] provides a reduced bound for the subscript n in (1.2).

Lemma 2.2. *Let M be a positive integer and p/q be a convergent of the continued fraction of the irrational number τ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\tau q\|$, where $\|.\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The following lemma will be used in our proof. It is seen in [4, Lemma 7].

Lemma 2.3. *Let $r \geq 1$ and $H > 0$ be such that $H > (4r^2)^r$ and $H > L/(\log L)^r$. Then*

$$L < 2^r H (\log H)^r.$$

The following result shows that n is larger than a in (1.2).

Lemma 2.4. *All solutions of (1.2) satisfy*

$$a\left(\frac{\log x}{\log\alpha}\right) - \frac{\log 2}{\log\alpha} + 1 < n < a\left(\frac{\log x}{\log\alpha}\right) + \frac{\log 3}{\log\alpha} + 2.$$

Proof. By virtue of (1.1) and (1.2), we have

$$\alpha^{n-2} < N_n = x^a \pm x^b + 1 < x^a + x^b + 1 < 3x^a.$$

Taking logarithm on both sides,

$$(n-2)\log\alpha < \log 3 + a\log x,$$

which implies

$$n < a\left(\frac{\log x}{\log\alpha}\right) + \frac{\log 3}{\log\alpha} + 2.$$

Similarly,

$$\frac{x^a}{2} < x^a - x^b < x^a \pm x^b + 1 = N_n < \alpha^{n-1}$$

gives

$$a\left(\frac{\log x}{\log \alpha}\right) - \frac{\log 2}{\log \alpha} + 1 < n.$$

□

The following result will also be used in our proof.

Lemma 2.5. *If $|e^z - 1| < y < \frac{1}{2}$ for real values of z and y, then $|z| < 2y$.*

Proof. If $z \geq 0$, we have $0 \leq z \leq e^z - 1 = |e^z - 1| < y$. If $z < 0$, then $|e^z - 1| < \frac{1}{2}$. From this, we get $e^{|z|} < 2$, and therefore $0 < |z| < e^{|z|} - 1 = e^{|z|}|e^z - 1| < 2y$. so, in both cases, we get $|z| < 2y$. □

3. Proof of Theorem 1.1

Using Binet's formula of Narayana's cows sequence in (1.2), we get

$$C_\alpha \alpha^{n+2} + C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2} = x^a \pm x^b + 1. \tag{3.1}$$

We can write (3.1)as

$$C_\alpha \alpha^{n+2} - x^a = -\left(C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2}\right) \pm x^b + 1,$$

which implies

$$\left|C_\alpha \alpha^{n+2} - x^a\right| < \frac{1}{2} + x^b + 1 < x^{b+1}.$$

Dividing both sides by x^a, we get

$$\left|C_\alpha \alpha^{n+2} x^{-a} - 1\right| < \frac{1}{x^{a-b-1}}. \tag{3.2}$$

Put

$$\Gamma = C_\alpha \alpha^{n+2} x^{-a} - 1.$$

One can check that $\Gamma \neq 0$. Suppose $\Gamma = 0$, then

$$C_\alpha \alpha^{n+2} = x^a. \tag{3.3}$$

Let σ be the automorphism of the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$ defined by $\sigma(\alpha) = \beta$, where $f(x) = x^3 - x^2 - 1$ is the minimal polynomial of α. Applying σ on both sides of (3.3), we get

$$|C_\beta \beta^{n+2}| = x^a,$$

which is not possible since $|C_\beta \beta^{n+2}| < |C_\beta| \approx 0.407506\ldots < 1$, whereas $x^a > 1$. Therefore, $\Gamma \neq 0$. Now, we are ready to apply Theorem 2.1 with the following data:

$$\eta_1 = C_\alpha,\ \eta_2 = \alpha,\ \eta_3 = x,\ b_1 = 1,\ b_2 = n+2,\ b_3 = -a,\ l = 3,$$

with $d_{\mathbb{L}} = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

Since $a < n$, take $D = n + 2$. The heights of η_1, η_2, η_3 are calculated as follows.

$$h(\eta_1) = h(C_\alpha) = \frac{\log 31}{3},\ h(\eta_2) = h(\alpha) = \frac{\log \alpha}{3},\ h(\eta_3) = h(x) = \log x.$$

Thus, we take

$$A_1 = \log 31,\ \ A_2 = \log \alpha,\ \ A_3 = 3 \log x.$$

Applying Theorem 2.1 we find

$$\begin{aligned}\log|\Gamma| &> -1.4 \cdot 30^6 3^{4.5} 3^2 (1 + \log 3)(1 + \log(n + 2))(\log 31)(\log \alpha)(3 \log x)\\ &> -1.1 \cdot 10^{13} \log(1 + \log(n + 2)) \log x.\end{aligned}$$

The above inequality together with (3.2), gives

$$(a - b - 1) \log x < 1.1 \cdot 10^{13} (\log x)(1 + \log(n + 2)).$$

Then, we get

$$(a - b) < 1.2 \cdot 10^{13} (1 + \log(n + 2)). \tag{3.4}$$

Writing (3.1) in another way,

$$C_\alpha \alpha^{n+2} - x^a \mp x^b = -\left(C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2}\right) + 1.$$

Taking absolute values on both sides, we obtain

$$\left|C_\alpha \alpha^{n+2} - x^a \mp x^b\right| < 1.5.$$

Dividing both sides by $x^a \pm x^b$, we get

$$\begin{aligned}\left|C_\alpha (x^{a-b} \pm 1)^{-1} \alpha^{n+2} x^{-b} - 1\right| &< \frac{1.5}{x^a \pm x^b}\\ &< \frac{3}{x^a}\\ &< \frac{9}{\alpha^{n-2}} < \frac{1}{\alpha^{n-6}}.\end{aligned} \tag{3.5}$$

Put

$$\Gamma' = C_\alpha (x^{a-b} \pm 1)^{-1} \alpha^{n+2} x^{-b} - 1.$$

By using similar reason as above we can show that $\Gamma' \neq 0$. Again with the notations of Theorem 2.1, we take

$$\eta_1 = C_\alpha (x^{a-b} \pm 1)^{-1},\ \ \eta_2 = \alpha,\ \ \eta_3 = x,\ \ b_1 = 1,\ \ b_2 = n + 2,\ \ b_3 = -b,\ \ l = 3,$$

with $d_{\mathbb{L}} = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ is 3.

Since $b < a < n$, take $D = n + 2$. The height of η_1 is computed as

$$\begin{aligned}h(\eta_1) &= h\left(C_\alpha (x^{a-b} \pm 1)^{-1}\right)\\ &\leq h(C_\alpha) + h\left(x^{a-b} \pm 1\right)\\ &\leq \frac{\log 31}{3} + (a - b) \log x + \log 2.\end{aligned}$$

Hence from (3.4), we get

$$h(\eta_1) < 1.21 \cdot 10^{13} (1 + \log(n + 2)) \log x.$$

The heights for η_2 and η_3 are

$$h(\eta_2) = \frac{\log \alpha}{3} \text{ and } h(\eta_3) = \log x.$$

So, we take

$$A_1 = 3.64 \cdot 10^{13}(1 + \log(n + 2)) \log x,\ A_2 = \log \alpha \text{ and } A_3 = 3 \log x.$$

Using all these values in Theorem 2.1, we have

$$\log |\Gamma'| > -1.4 \cdot 30^6 3^{4.5} 3^2 (1 + \log 3)(1 + \log(n + 2))(3.64 \cdot 10^{13}(1 + \log(n + 2)) \log x) \\ (\log \alpha)(3 \log x).$$

Comparing the above inequality with (3.5) implies that

$$(n - 6) \log \alpha < 1.2 \cdot 10^{26}(1 + \log(n + 2))^2 (\log x)^2. \tag{3.6}$$

Thus, we conclude that

$$n < 3.2 \cdot 10^{26}(1 + \log(n + 2))^2 (\log x)^2 < 5.1 \cdot 10^{27} (\log n)^2 (\log x)^2. \tag{3.7}$$

With the notation of Lemma 2.3, we take $r = 2$, $L = n$ and $H = 5.1 \cdot 10^{27}(\log x)^2$. Applying Lemma 2.3, we have

$$\begin{aligned} n &< 2^2(5.1 \cdot 10^{27}(\log x)^2)(\log(5.1 \cdot 10^{27}(\log x)^2)^2 \\ &< (2.1 \cdot 10^{28}(\log x)^2)(64 + 2 \log \log x)^2 \\ &< (2.1 \cdot 10^{28}(\log x)^2)(92 \log x)^2 \\ &< 1.8 \cdot 10^{32}(\log x)^4. \end{aligned}$$

Now, for $2 \le x \le 30$, we have

$$n < 2.4 \cdot 10^{34}.$$

This bound is too large, so we need to reduce it. Put

$$\Lambda = (n + 2) \log \alpha - a \log x + \log C_\alpha.$$

The inequality (3.2) can be written as

$$|e^\Lambda - 1| < \frac{1}{x^{a-b-1}} < \frac{1}{2^{a-b-1}}.$$

Notice that $\Lambda \neq 0$ as $e^\Lambda - 1 = \Gamma \neq 0$. Assuming $(a - b) \ge 2$, the right-hand side in the above inequality is at most $\frac{1}{2}$. Using Lemma 2.5 we obtain

$$0 < |\Lambda| < \frac{2}{x^{a-b-1}},$$

which implies that

$$|(n + 2) \log \alpha - a \log x + \log C_\alpha| < \frac{2}{x^{a-b-1}}.$$

Dividing both sides by $\log x$ gives

$$\left| n \left(\frac{\log \alpha}{\log x} \right) - a + \left(\frac{\log(\alpha^2 C_\alpha)}{\log x} \right) \right| < \frac{2.89}{x^{a-b-1}}. \tag{3.8}$$

Now, we are ready to apply Lemma 2.2 with the following data:

$$u = n,\ \tau = \left(\frac{\log \alpha}{\log x} \right),\ v = a,\ \mu = \left(\frac{\log(\alpha^2 C_\alpha)}{\log x} \right),\ A = 2.89,\ B = x,\ w = a - b - 1.$$

Note that $\frac{\log \alpha}{\log x} \notin \mathbb{Q}$ because if $\frac{\log \alpha}{\log x} = \frac{s}{t}$ for some coprime positive integers s and t, we would have $x^s = \alpha^t \in \mathbb{Z}$. Using the automorphism σ defined above, we get $1 < x^s = |\beta^t| < 1$, a contradiction. Choosing $M = 2.4 \cdot 10^{34}$, we find the convergent q_{84} exceeds $6M$ with

$\varepsilon := \|\mu q_{84}\| - M\|\tau q_{84}\| > 0.015$. So all the conditions of Lemma 2.2 are fulfilled. Hence, there exists no solution to the inequality (3.8) if

$$a - b - 1 \geq \frac{\log((2.89 q_{84})/0.015)}{\log x} \geq 125.$$

Thus, we must have $a - b < 126$.

Now for $a - b < 126$, put

$$\Lambda' = (n+2)\log\alpha - b\log x + \log\left(\frac{C_\alpha}{x^{a-b} \pm 1}\right).$$

The inequality (3.5) can be written as

$$|e^{\Lambda'} - 1| < \frac{1}{\alpha^{n-6}}.$$

Assuming $n \geq 8$, the right-hand side in the above inequality is at most $\frac{1}{\alpha^2} < \frac{1}{2}$. Using Lemma 2.5 we get

$$|\Lambda'| < \frac{2}{\alpha^{n-6}},$$

which implies that

$$\left|(n+2)\log\alpha - b\log x + \log\left(\frac{C_\alpha}{x^{a-b} \pm 1}\right)\right| < \frac{2}{\alpha^{n-6}}.$$

Dividing both sides by $\log x$ gives

$$\left| n\left(\frac{\log\alpha}{\log x}\right) - b + \left(\frac{\log\left(\frac{\alpha^2 C_\alpha}{x^{a-b} \pm 1}\right)}{\log x}\right)\right| < 2.89 \cdot \alpha^{-(n-6)}. \tag{3.9}$$

Now, let

$$u = n,\ \tau = \left(\frac{\log\alpha}{\log x}\right),\ v = b,\ \mu = \left(\frac{\log\left(\frac{\alpha^2 C_\alpha}{x^{a-b} \pm 1}\right)}{\log x}\right)$$
$$A = 2.89,\ B = \alpha,\ w = n - 6.$$

We find q_{84} exceeds $6M$ with $\varepsilon := \|\mu q_{84}\| - M\|\tau q_{84}\| > 0.015$. Thus, by Lemma 2.2, we see that the inequality (3.9) has no solution if

$$n \geq \frac{\log((2.89 \cdot q_{84})/0.015)}{\log\alpha} \geq 329.$$

So, it has to be $n < 329$. Finally, we run a program in *Mathematica* with $2 \leq x \leq 30$ and $n < 329$ and get all the solutions listed in Theorem 1.1.

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