THE SELF-COUNTING FLOW

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Abstract. This paper is based on the article "The self-counting identity", published in the *Fibonacci Quarterly* in May 2017, vol. 55 and can be considered as its continuation.

In the beginning, we define the "self-counting flow Φ", which represents a tool for getting from one positive integer sequence to a corresponding other one. It is - so to say - a flow on all positive integer sequences and thereby the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$ shows itself as a unique fixed point.

Various methods allow us to study the properties of the flow Φ such as its trajectories and the attraction of its fixed point. We also examine whether the self-counting sequence $\{a_k\}_{k=1}^{\infty}$ is the point of convergence of each positive integer sequence under a repeated application of the self-counting flow Φ.

At the end of this article, we show some properties of other flows on positive integer sequences, for example those of the "Fibonacci flow $\mathcal{F}$".

1. Introduction

This paper is based on the talk "The Self-Counting Flow" which we gave at "The 20$^{\text{th}}$ International Conference on Fibonacci Numbers and Their Applications, July 25-29, 2022" in Sarajevo. The first part of this article was presented there and the second part is some additional material.

First, we introduce the self-counting flow Φ, which is a flow on the positive integer sequences, then we study and prove its most important properties. We also give some extensions and generalizations of the self-counting flow Φ at the end of the paper.

The main result of the paper is the proof that all positive integer sequences get closer and closer to the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$ under a repeated application of the self-counting flow Φ.

The basic ideas and notions of this paper are inspired by the book "Hyperbolic Flows" [1], which was the main literature for a course at the ETH Zurich attended by the author. In this book [1], all notions of this article are introduced in the same or a similar way, such as a flow, the boundary of a flow, the fixed points and the cyclic points of flow or the trajectories of a point under a flow. The main difference is that the flows introduced in this paper are not hyperbolic flows and have nothing to do with them. The presented flows in this paper are flows on the integer sequences.

2. Definitions and Basic Facts

Let $b_k \in \mathbb{N} := \{1, 2, 3, 4, \ldots\}$ for all $k \in \mathbb{N}$. We let

$$\{b_k\}_{k=1}^{\infty} = \{b_1, b_2, b_3, b_4, \ldots\} \in \mathbb{N}^{\mathbb{N}}$$

denote a positive integer sequence.

Definition 2.1 (The concatenation $\times$ and the product $\prod$ of integer sequences)**.** *We define for all $n \in \mathbb{N}$ the concatenation $\times$ of a finite integer sequence $\{b_k\}_{k=1}^{n}$ with the infinite integer* *sequence* $\{c_k\}_{k=1}^{\infty}$ *by*

$$\begin{aligned}\{b_k\}_{k=1}^{n} \times \{c_k\}_{k=1}^{\infty} &= \{b_1, b_2, b_3, \ldots, b_n\} \times \{c_1, c_2, c_3, \ldots\}\\ &= \{b_1, b_2, b_3, \ldots, b_n, c_1, c_2, c_3, \ldots\}.\end{aligned}$$

An equation of the form

$$\{c_k\}_{k=1}^{\infty} = \{b_k\}_{k=1}^{n} \times \mathbb{N}^{\mathbb{N}} = \{b_1, b_2, b_3, \ldots, b_{n-1}, b_n, c_{n+1}, c_{n+2}, \ldots\}$$

means that $\{c_k\}_{k=1}^{\infty}$ *is any number sequence that begins with* $\{b_k\}_{k=1}^{n} = \{b_1, b_2, b_3, \ldots, b_{n-1}, b_n\}$. *Let the numbers* n_1, n_2, n_3, $\ldots$ *be an infinite sequence of natural numbers. We define the product* $\prod$ *of the infinitely many integer sequences*

$$\left\{b_1^{(1)}, b_2^{(1)}, b_3^{(1)}, \ldots, b_{n_1}^{(1)}\right\},\ \left\{b_1^{(2)}, b_2^{(2)}, b_3^{(2)}, \ldots, b_{n_2}^{(2)}\right\},\ \left\{b_1^{(3)}, b_2^{(3)}, b_3^{(3)}, \ldots, b_{n_3}^{(3)}\right\}, \ldots$$

by

$$\begin{aligned}&\prod_{k=1}^{\infty}\left\{b_1^{(k)}, b_2^{(k)}, b_3^{(k)}, \ldots, b_{n_k}^{(k)}\right\}\\ &= \left\{b_1^{(1)}, b_2^{(1)}, b_3^{(1)}, \ldots, b_{n_1}^{(1)}\right\} \times \left\{b_1^{(2)}, b_2^{(2)}, b_3^{(2)}, \ldots, b_{n_2}^{(2)}\right\} \times \left\{b_1^{(3)}, b_2^{(3)}, b_3^{(3)}, \ldots, b_{n_3}^{(3)}\right\} \times \ldots\\ &= \left\{\textit{all possible sequences } \{c_1, c_2, c_3, c_4, \ldots\},\ c_k \in \left\{b_1^{(k)}, b_2^{(k)}, b_3^{(k)}, \ldots, b_{n_k}^{(k)}\right\} \textit{ for all } k \in \mathbb{N}\right\}.\end{aligned}$$

Moreover, we denote by $\lfloor x \rfloor$ the floor of x and by $\lceil x \rceil$ the ceilling of x. An expression of the form $d|n$ means that "d divides n" and $d \nmid n$ means that "d does not divide n".

The function $\mathbb{I}_{d|n}$ is the indicator function of the event that $d|n$, that means

$$\mathbb{I}_{d|n} := \begin{cases} 1, & \text{if } d|n,\\ 0, & \text{if } d \nmid n.\end{cases}$$

As usual, the symbol $\delta_{x,y}$ denotes the Kronecker-Delta-Symbol, which is defined by

$$\delta_{x,y} : = \begin{cases} 1, & \text{if } x = y,\\ 0, & \text{if } x \neq y.\end{cases}$$

The function $\operatorname{sign}(x)$ denotes the signum function given by

$$\operatorname{sign}(x) : = \begin{cases} 1, & \text{if } x > 0,\\ 0, & \text{if } x = 0,\\ -1, & \text{if } x < 0.\end{cases}$$

Definition 2.2 (The self-counting sequence, [2], [3], [4])**.** *Let the integer sequence*

$$\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\} \in \mathbb{N}^{\mathbb{N}}$$

be the self-counting sequence. The k*-th term* a_k *of this sequence is given explicitly by*

$$\begin{aligned}a_k &= \left\lfloor \frac{1}{2} + \sqrt{2k} \right\rfloor\\ &= \left\lceil \frac{1}{2}\left(\sqrt{8k+1} - 1\right)\right\rceil.\end{aligned}$$

The self-counting identity [2] states for all $x \in \mathbb{C}$ with $|x| < 1$ that

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{a_k}} = \sum_{k=1}^{\infty} a_k x^k. \tag{2.1}$$

3. The Self-Counting Flow Φ

In this section, we present the self-counting flow Φ.

3.1. Definition of the Self-Counting Flow Φ.

In this subsection, we introduce the definition of the self-counting flow Φ. From now on, all power series will be considered as formal power series, unless otherwise specified. We make the following definition.

Definition 3.1 (A flow Δ on $\mathbb{N}^{\mathbb{N}}$, $\mathbb{Z}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$)**.**
A map $\Delta : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, which maps every sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ to some sequence $\{c_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ is called a flow on $\mathbb{N}^{\mathbb{N}}$.
A map $\Delta : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$, which maps every sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{Z}^{\mathbb{N}}$ to some sequence $\{c_k\}_{k=1}^{\infty} \in \mathbb{Z}^{\mathbb{N}}$ is called a flow on $\mathbb{Z}^{\mathbb{N}}$.
A map $\Delta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, which maps every sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ to some sequence $\{c_k\}_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ is called a flow on $\mathbb{R}^{\mathbb{N}}$.

If we apply a flow Δ repeatedly to some point $\{b_k\}_{k=1}^{\infty}$, we get a trajectory starting at $\{b_k\}_{k=1}^{\infty}$ into the spaces $\mathbb{N}^{\mathbb{N}}$, $\mathbb{Z}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$. We think of it as "the point $\{b_k\}_{k=1}^{\infty}$ flowing into the spaces $\mathbb{N}^{\mathbb{N}}$, $\mathbb{Z}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$". The study of these trajectories can be interesting.

Definition 3.2 (The self-counting flow Φ)**.**
From a given positive integer sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$, we get the new positive integer sequence $\{c_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$, that fulfills

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{b_k}} = \sum_{k=1}^{\infty} c_k x^k.$$

This formula describes therefore a flow Φ on all positive integer sequences $\mathbb{N}^{\mathbb{N}}$. We call this flow the self-counting flow Φ and define it by

$$\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$$

$$\{b_k\}_{k=1}^{\infty} \mapsto \{c_k\}_{k=1}^{\infty} \quad \text{if} \quad \sum_{k=1}^{\infty} \frac{x^k}{1 - x^{b_k}} = \sum_{k=1}^{\infty} c_k x^k.$$

Using the self-counting flow Φ, we write for this

$$\Phi\left(\{b_k\}_{k=1}^{\infty}\right) = \{c_k\}_{k=1}^{\infty}.$$

We have the following theorem.

Theorem 3.3 (Explicit formula for $\{c_k\}_{k=1}^{\infty} = \Phi\left(\{b_k\}_{k=1}^{\infty}\right)$)**.**
We have for the sequence $\{c_k\}_{k=1}^{\infty} = \Phi\left(\{b_k\}_{k=1}^{\infty}\right) \in \mathbb{N}^{\mathbb{N}}$ the explicit formula

$$c_k = \sum_{n=1}^{k} \mathbb{I}_{b_n | (k-n)}, \tag{3.1}$$

if a sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ *is given. This formula is also equivalent to the expression*

$$c_k = 1 + \sum_{n=1}^{k-1} \mathbb{I}_{b_n|(k-n)}. \tag{3.2}$$

Proof. We can calculate that

$$\begin{aligned}\sum_{k=1}^{\infty} c_k x^k &= \sum_{k=1}^{\infty} \frac{x^k}{1-x^{b_k}} \\ &= \sum_{k=1}^{\infty} x^k \sum_{m=0}^{\infty} x^{m\cdot b_k} \\ &= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} \mathbb{I}_{b_n|(k-n)} \right) x^k.\end{aligned}$$

Equating coefficients on both sides, we obtain the claimed explicit formula (3.1). The second formula (3.2) follows, because $b_n|0$ for all $n \in \mathbb{N}$. □

The self-counting identity (2.1) implies that

$$\begin{aligned}\Phi\left(\{a_k\}_{k=1}^{\infty}\right) &= \Phi\left(\{1,2,2,3,3,3,4,4,4,4,\ldots\}\right) \\ &= \{1,2,2,3,3,3,4,4,4,4,\ldots\} \\ &= \{a_k\}_{k=1}^{\infty}\end{aligned}$$

and therefore, the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1,2,2,3,3,3,4,4,4,4,\ldots\}$ is a fixed point of the self-counting flow Φ.

This implies with $\{b_k\}_{k=1}^{\infty} := \{a_k\}_{k=1}^{\infty}$ that $\{c_k\}_{k=1}^{\infty} = \Phi\left(\{b_k\}_{k=1}^{\infty}\right) = \{a_k\}_{k=1}^{\infty}$ and we obtain with the explicit formula (3.1), as well as with the expression (3.2) the following corollary.

Corollary 3.4 (The self-counting sequence divisor identity)**.**
We have for the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1,2,2,3,3,3,4,4,4,4,\ldots\}$ *the identity*

$$a_k = \sum_{n=1}^{k} \mathbb{I}_{a_n|(k-n)}. \tag{3.3}$$

This formula is also equivalent to the explicit expression

$$a_k = 1 + \sum_{n=1}^{k-1} \mathbb{I}_{a_n|(k-n)}. \tag{3.4}$$

3.2. **The Reason for the Choice of the Domain $\mathbb{N}^{\mathbb{N}}$ in the Definition of Φ.** Because of the definition of the self-counting flow Φ via the formula

$$\Phi : \{b_k\}_{k=1}^{\infty} \to \{c_k\}_{k=1}^{\infty} \text{ if } \sum_{k=1}^{\infty} \frac{x^k}{1-x^{b_k}} = \sum_{k=1}^{\infty} c_k x^k,$$

we must have that

- $b_k \neq 0$ for all $k \in \mathbb{N}$ (otherwise we divide by 0)
- $b_k \in \mathbb{Z}$ for all $k \in \mathbb{N}$ (otherwise we have non-integer exponents).

We also have that

$$\Phi\left(\{-1,-1,-1,-1,\ldots\}\right) = \{0,-1,-2,-3,-4,-5,-6,-7,-8,-9,-10,\ldots\},$$

because

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x^{-1}} = \frac{x}{(1-x^{-1})(1-x)} = -\frac{x^2}{(1-x)^2} = -x^2 - 2x^3 - 3x^4 - 4x^5 - \ldots$$

and that

$$\Phi\left(\{1,-1,-1,-1,\ldots\}\right) = \{1,1,0,-1,-2,-3,-4,-5,-6,-7,-8,\ldots\},$$

because

$$\frac{x}{1-x} + \sum_{k=2}^{\infty} \frac{x^k}{1-x^{-1}} = -\frac{x(x^2+x-1)}{(1-x)^2} = x + x^2 - x^4 - 2x^5 - 3x^6 - \ldots.$$

Therefore

- an iterative application of the self-counting flow Φ is only always possible if $b_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ (otherwise we would get 0's at least for some $\{b_k\}_{k=1}^{\infty}$'s)
- $\Phi\left(\mathbb{N}^{\mathbb{N}}\right) \subset \mathbb{N}^{\mathbb{N}}$, because $\frac{x^k}{1-x^{b_k}} = x^k + \ldots$ if $b_k \in \mathbb{N}$ for all $k \in \mathbb{N}$.

This shows that

- the natural domain for a repeated application of the self-counting flow Φ is $\mathbb{N}^{\mathbb{N}}$
- Φ is well-defined on $\mathbb{N}^{\mathbb{N}}$.

3.3. **An Example of a Calculation with the Self-Counting Flow Φ.** In this subsection, we will show by an example that successive application of Φ gives sequences closer and closer to the self-counting sequence $\{a_k\}_{k=1}^{\infty}$. We start with the point

$$\{1,1,1,1,\ldots\} \in \mathbb{N}^{\mathbb{N}}$$

in the space $\mathbb{N}^{\mathbb{N}}$ of all positive integer sequences. Then, we have that

$$\Phi\left(\{1,1,1,1,\ldots\}\right) = \{1,2,3,4,5,6,7,8,9,10,\ldots\} = \{k\}_{k=1}^{\infty},$$

because

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x} = \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots.$$

If we now apply the self-counting flow Φ again to the point $\Phi(\{1,1,1,1,\ldots\}) = \{k\}_{k=1}^{\infty}$, we obtain

$$\begin{aligned}\Phi\left(\{k\}_{k=1}^{\infty}\right) &= \Phi\left(\{1,2,3,4,5,6,7,8,9,10,\ldots\}\right)\\ &= \{1,2,2,3,2,4,2,4,3,4,\ldots\}\\ &= \{\tau(k)\}_{k=1}^{\infty},\end{aligned}$$

where $\tau(k) = \sum_{d|k} 1 = \sum_{d=1}^{k} \mathbb{I}_{d|k}$ is the divisor function which gives the number of positive divisors of the number $k \in \mathbb{N}$. This is true, because of the well-known Lambert type identity [5, Theorem 310, p. 258]

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x^k} = \sum_{k=1}^{\infty} \tau(k) x^k$$

for $x \in \mathbb{C}$ with $|x| < 1$.

We can make a further transformation with Φ to get

$$\begin{aligned}\Phi\left(\{\tau(k)\}_{k=1}^{\infty}\right) &= \Phi\left(\{1,2,2,3,2,4,2,4,3,4,\ldots\}\right)\\ &= \{1,2,2,3,3,3,5,3,5,5,\ldots\}\\ &= \{\tau_1(k)\}_{k=1}^{\infty},\end{aligned}$$

because for $x \in \mathbb{C}$ with $|x| < 1$ it holds

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{\tau(k)}} = \sum_{k=1}^{\infty} \tau_1(k) x^k,$$

where $\tau_1(k) = \sum_{d=1}^{k} \mathbb{I}_{\tau(d)|(k-d)}$.

Applying the self-counting flow Φ one more time, we get

$$\begin{aligned}\Phi\left(\{\tau_1(k)\}_{k=1}^{\infty}\right) &= \Phi\left(\{1,2,2,3,3,3,5,3,5,5,\ldots\}\right)\\ &= \{1,2,2,3,3,3,4,4,4,4,5,5,4,6,5,5,\ldots\}\\ &= \{\tau_2(k)\}_{k=1}^{\infty},\end{aligned}$$

where $\tau_2(k) = \sum_{d=1}^{k} \mathbb{I}_{\tau_1(d)|(k-d)}$. And so on.

4. The Main Properties of the Self-Counting Flow Φ

In this section, we prove the main properties of the self-counting flow Φ.

4.1. The Unique Fixed Point of the Self-Counting Flow Φ.

We have the following fixed point theorem.

Theorem 4.1 (The unique fixed point of the self-counting flow Φ)**.**
The self-counting flow Φ has a unique fixed point given by the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1,2,2,3,3,3,4,4,4,4,\ldots\}$.

Proof. Because of the self-counting identity (2.1)

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{a_k}} = \sum_{k=1}^{\infty} a_k x^k,$$

we already know that the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1,2,2,3,3,3,4,4,4,4,\ldots\}$ is a fixed point of the self-counting flow Φ. So we only need to prove its uniqueness.

The fixed point condition is that

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{b_k}} = \sum_{k=1}^{\infty} b_k x^k.$$

If we want that

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{b_k}} = \frac{x}{1 - x^{b_1}} + O(x^2) = b_1 x + O(x^2),$$

we need that $b_1 = 1$. From now on, we have no choices for the coefficients b_k with $k \in \mathbb{N}_{\geq 2}$, because

$$\sum_{k=1}^{n} \frac{x^k}{1 - x^{b_k}} = \sum_{k=1}^{n} b_k x^k + (b_{n+1} - 1)x^{n+1} + O(x^{n+2}) \quad \text{for all } n \in \mathbb{N}.$$

Here the coefficient $b_{n+1}-1$ appears, because of the fixed point condition and the relations

$$\frac{x^{n+1}}{1-x^{b_{n+1}}} = x^{n+1} + O(x^{n+2}),$$
$$\frac{x^k}{1-x^{b_k}} = O(x^{n+2}) \text{ for all } k \geq n+2.$$

Therefore, there is at most one fixed point $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$. □

4.2. **The Trajectories under the Self-Counting Flow Φ.** In this subsection, we study the trajectories of the self-counting flow Φ.

Theorem 4.2 (The trajectories under the self-counting flow Φ)**.**
The self-counting sequence $\{a_k\}_{k=1}^{\infty}$ is an attractive fixed point of the self-counting flow Φ. Every number sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ converges under the self-counting flow Φ to the self-counting sequence $\{a_k\}_{k=1}^{\infty}$, that is

$$\lim_{n\to\infty} \left(\Phi^n\left(\{b_k\}_{k=1}^{\infty}\right)\right) = \{a_k\}_{k=1}^{\infty} = \{1,2,2,3,3,3,4,4,4,4,\ldots\}$$

for all number sequences $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$, where

$$\Phi^n\left(\{b_k\}_{k=1}^{\infty}\right) := \underbrace{\Phi \circ \Phi \circ \Phi \circ \cdots \circ \Phi}_{n-times}\left(\{b_k\}_{k=1}^{\infty}\right).$$

Here the convergence of an integer sequence $\{b_k\}_{k=1}^{\infty}$ under the self-counting flow Φ to an integer sequence $\{c_k\}_{k=1}^{\infty}$ means that the point $\Phi^n(\{b_k\}_{k=1}^{\infty})$ gets closer and closer to the integer sequence $\{c_k\}_{k=1}^{\infty}$ if $n \to \infty$. This means that for every $m \in \mathbb{N}$, there exists $d_m \in \mathbb{N}$ such that the first m coefficients of $\Phi^n(\{b_k\}_{k=1}^{\infty})$ coincide with the first m coefficients of $\{c_k\}_{k=1}^{\infty}$ if $n \geq d_m$.

Proof. We will prove that $\Phi^n(\{b_k\}_{k=1}^{\infty})$ begins with the first n elements of the self-counting sequence $\{a_k\}_{k=1}^{\infty}$. We have for any sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ that

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x^{b_k}} = x + O(x^2) = c_1 x + O(x^2).$$

Therefore, we get for $\Phi(\{b_k\}_{k=1}^{\infty}) = \{c_k\}_{k=1}^{\infty}$ that $c_1 = a_1 = 1$. If it holds that

$$\{b_k\}_{k=1}^{\infty} = \{a_k\}_{k=1}^{n} \times \mathbb{N}^{\mathbb{N}} = \{1,2,2,\ldots,a_{n-1},a_n,b_{n+1},b_{n+2},\ldots\},$$

where $\{a_k\}_{k=1}^{\infty} = \{1,2,2,3,3,3,4,4,4,4,\ldots\}$ is the self-counting sequence, then we have that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{x^k}{1-x^{b_k}} &= \sum_{k=1}^{n} \frac{x^k}{1-x^{a_k}} + \frac{x^{n+1}}{1-x^{b_{n+1}}} + O(x^{n+2}) \\
&= \sum_{k=1}^{n} a_k x^k + (a_{n+1}-1)x^{n+1} + x^{n+1} + O(x^{n+2}) \\
&= \sum_{k=1}^{n+1} a_k x^k + O(x^{n+2}).
\end{aligned}$$

Therefore, it holds for $\{b_k\}_{k=1}^{\infty} = \{a_k\}_{k=1}^{n} \times \mathbb{N}^{\mathbb{N}}$ that

$$\Phi(\{b_k\}_{k=1}^{\infty}) = \{a_k\}_{k=1}^{n+1} \times \mathbb{N}^{\mathbb{N}} = \{1,2,2,\ldots,a_n,a_{n+1},c_{n+2},c_{n+3},\ldots\}$$

and this implies that $\lim_{n\to\infty}\left(\Phi^n(\{b_k\}_{k=1}^{\infty})\right) = \{a_k\}_{k=1}^{\infty}$ for all $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$. □

4.3. **The Cyclic Point of the Self-Counting Flow Φ.** We define cyclic points or circular points of the self-counting flow Φ by the following definition.

Definition 4.3 (Cyclic points of the self-counting flow Φ)**.**
A cyclic point $\{b_k\}_{k=1}^{\infty}$ of the self-counting flow Φ is a point $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ such that we have $\Phi^n\left(\{b_k\}_{k=1}^{\infty}\right) = \{b_k\}_{k=1}^{\infty}$ for some $n \in \mathbb{N}$.

Corollary 4.4 (The unique cyclic point $\{a_k\}_{k=1}^{\infty}$ of the self-counting flow Φ)**.**
The self-counting flow Φ has a unique cyclic point given by the self-counting sequence $\{a_k\}_{k=1}^{\infty}$.

Proof. Because the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, \ldots\}$ is the unique attractive fixed point of the self-counting flow Φ and all sequences in $\mathbb{N}^{\mathbb{N}}$ converge under this flow Φ to $\{a_k\}_{k=1}^{\infty}$, there cannot exist any other cyclic points besides $\{a_k\}_{k=1}^{\infty}$. ☐

4.4. **The Boundary of the Self-Counting Flow Φ.** In this subsection, we study the points of $\mathbb{N}^{\mathbb{N}}$, which are not contained in the image of the self-counting flow Φ. The following definition makes this idea precise.

Definition 4.5 (The boundary of the self-counting flow Φ)**.**
The boundary $\mathcal{B}_\Phi := \mathbb{N}^{\mathbb{N}} \setminus \Phi\left(\mathbb{N}^{\mathbb{N}}\right)$ of the self-counting flow Φ on $\mathbb{N}^{\mathbb{N}}$ consists of the points $\{c_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ such that there does not exist a point $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ with $\Phi\left(\{b_k\}_{k=1}^{\infty}\right) = \{c_k\}_{k=1}^{\infty}$. That means that the points of $\mathcal{B}_\Phi$ are not in the image of the self-counting flow Φ.

The structure of the boundary $\mathcal{B}_\Phi$ is complicated, but we know the following theorem.

Theorem 4.6 (The self-counting flow Φ is not boundary-free)**.**
The boundary $\mathcal{B}_\Phi$ is non-empty, that is $\mathcal{B}_\Phi \neq \emptyset$.

Proof. It suffices to find one boundary point contained in $\mathcal{B}_\Phi$. We have for example the boundary point

$$\{1, 1, 1, 1, \ldots\},$$

because there does not exist a point $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ such that

$$\Phi\left(\{b_k\}_{k=1}^{\infty}\right) = \{1, 1, 1, 1, \ldots\}.$$

This is because

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{b_k}} &= \frac{x}{1 - x^{b_1}} + \sum_{k=2}^{b_1} \frac{x^k}{1 - x^{b_k}} + \frac{x^{b_1+1}}{1 - x^{b_{b_1+1}}} + \sum_{k=b_1+2}^{\infty} \frac{x^k}{1 - x^{b_k}} \\ &= x + \ldots + 2x^{b_1+1} + \ldots.\end{aligned}$$

☐

Theorem 4.7 (Substructure of the boundary $\mathcal{B}_\Phi$ of the self-counting flow Φ)**.**
We have that

$$\mathbb{N}_{\geq 2} \times \mathbb{N}^{\mathbb{N}} \subset \mathcal{B}_\Phi.$$

Proof. This follows from the observation

$$\Phi\left(\mathbb{N}^{\mathbb{N}}\right) \subset \{1\} \times \mathbb{N}^{\mathbb{N}},$$

which follows from $c_1 = 1$ if

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{b_k}} = \sum_{k=1}^{\infty} c_k x^k \quad \text{with} \quad \{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}.$$

This fact was already proved in the proof of Theorem 4.2. □

Theorem 4.8 (Another boundary component)**.**
Let the set M be defined by

$$M := \prod_{k=1}^{\infty}\{1,\ldots,k\} = \{1\}\times\{1,2\}\times\{1,2,3\}\times\{1,2,3,4\}\times\ldots.$$

Then we have that

$$\mathbb{N}^{\mathbb{N}}\setminus M \subset \mathcal{B}_{\Phi}.$$

Proof. This follows from $\Phi(\mathbb{N}^{\mathbb{N}})\subset M$, which holds because from the explicit formula (3.1), we get that

$$1 \le c_k = \sum_{n=1}^{k}\mathbb{I}_{b_n|(k-n)} \le k$$

for all $k\in\mathbb{N}$. □

4.5. **The Non-Invertibility of the Self-Counting Flow Φ.** We have the following theorem.

Theorem 4.9 (The non-existence of the map Φ^{-1})**.**
The map Φ^{-1} does not exist, because for some sequences $\{c_k\}_{k=1}^{\infty}\in\Phi(\mathbb{N}^{\mathbb{N}})$ there exist two sequences $\{b_k\}_{k=1}^{\infty}\in\mathbb{N}^{\mathbb{N}}$ and $\{b_k^*\}_{k=1}^{\infty}\in\mathbb{N}^{\mathbb{N}}$, such that

$$\Phi(\{b_k\}_{k=1}^{\infty}) = \Phi(\{b_k^*\}_{k=1}^{\infty}) = \{c_k\}_{k=1}^{\infty}.$$

Moreover, there exist infinitely many pairs of sequences $\{b_k\}_{k=1}^{\infty}\in\mathbb{N}^{\mathbb{N}}$ and $\{b_k^*\}_{k=1}^{\infty}\in\mathbb{N}^{\mathbb{N}}$, such that

$$\Phi(\{b_k\}_{k=1}^{\infty}) = \Phi(\{b_k^*\}_{k=1}^{\infty}).$$

Proof. We have the identity

$$\frac{x}{1-x^2}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^2} = \frac{x}{1-x^4}+\frac{x^2}{1-x}+\frac{x^3}{1-x^4}, \tag{4.1}$$

which holds because

$$\begin{aligned}\frac{x}{1-x^2}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^2} &= \frac{x+x^2+x^3}{1-x^2}\\ &= \frac{(x+x^2+x^3)(1+x^2)}{1-x^4}\\ &= \frac{x+x^2+2x^3+x^4+x^5}{1-x^4}\\ &= \frac{x}{1-x^4}+\frac{x^2(1+x+x^2+x^3)}{1-x^4}+\frac{x^3}{1-x^4}\\ &= \frac{x}{1-x^4}+\frac{x^2}{1-x}+\frac{x^3}{1-x^4}.\end{aligned}$$

From this identity (4.1), we deduce for any $m\in\mathbb{N}$ that

$$\frac{x}{1-x^2}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^2}+\sum_{k=4}^{\infty}\frac{x^k}{1-x^m} = \frac{x}{1-x^4}+\frac{x^2}{1-x}+\frac{x^3}{1-x^4}+\sum_{k=4}^{\infty}\frac{x^k}{1-x^m},$$

which implies that

$$\Phi(\{2,2,2,m,m,m,m,m,\ldots\}) = \Phi(\{4,1,4,m,m,m,m,m,\ldots\}).$$

Therefore, we have for

$$\{b_k\}_{k=1}^{\infty} := \{2,2,2,m,m,m,m,m,\ldots\} \quad \text{and} \quad \{b_k^*\}_{k=1}^{\infty} := \{4,1,4,m,m,m,m,m,\ldots\}$$

that

$$\Phi(\{b_k\}_{k=1}^{\infty}) = \Phi(\{b_k^*\}_{k=1}^{\infty}),$$

which implies that the map Φ^{-1} does not exist. □

Theorem 4.10 (Special divisor structures identity)**.**
If it holds for two sequences $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ and $\{b_k^*\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ that

$$\{c_k\}_{k=1}^{\infty} = \Phi(\{b_k\}_{k=1}^{\infty}) = \Phi(\{b_k^*\}_{k=1}^{\infty}),$$

then we have for all $k \in \mathbb{N}$ that

$$c_k = \sum_{n=1}^{k} \mathbb{I}_{b_n|(k-n)} = \sum_{n=1}^{k} \mathbb{I}_{b_n^*|(k-n)}.$$

In particular, this formula holds for all sequences

$\{b_k\}_{k=1}^{\infty} = \{2,2,2,m,m,m,m,m,\ldots\}$ *and* $\{b_k^*\}_{k=1}^{\infty} = \{4,1,4,m,m,m,m,m,\ldots\}$ *with* $m \in \mathbb{N}$

defined in the proof of the above Theorem 4.9.

Proof. This theorem follows directly by applying the formula (3.1) to the equation

$$\{c_k\}_{k=1}^{\infty} = \Phi\left(\{b_k\}_{k=1}^{\infty}\right) = \Phi\left(\{b_k^*\}_{k=1}^{\infty}\right).$$

□

Theorem 4.11 (Pre-image of the point $\{1,2,3,4,5,6,7,8,\ldots\}$ under Φ)**.**
The point $\{1,1,1,1,\ldots\} \in \mathbb{N}^{\mathbb{N}}$ is the only point that maps under the self-counting flow Φ to the point $\{k\}_{k=1}^{\infty} = \{1,2,3,4,5,6,7,8,\ldots\} \in \mathbb{N}^{\mathbb{N}}$, that is:

$$\textit{If } \Phi\left(\{b_k\}_{k=1}^{\infty}\right) = \{k\}_{k=1}^{\infty} = \{1,2,3,4,5,6,7,8,\ldots\}, \ \textit{then } \{b_k\}_{k=1}^{\infty} = \{1,1,1,1,\ldots\}.$$

This implies that "the map Φ^{-1} exists at least for some points", which means that some points have only one point as a pre-image.

Proof. Because of the identity

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x} = (x+x^2+x^3+x^4+\ldots) + (x^2+x^3+x^4+\ldots) + (x^3+x^4+\ldots) + \ldots = \sum_{k=1}^{\infty} kx^k,$$

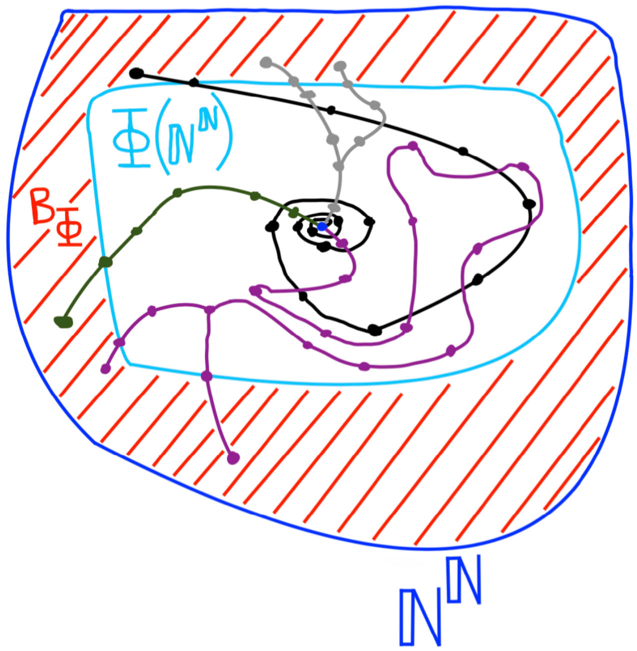
any other expression of the form

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x^{b_k}} \quad \text{with at least one } b_k \geq 2$$

cannot be equal to

$$\begin{aligned}\sum_{k=1}^{\infty} kx^k &= x + 2x^2 + 3x^3 + 4x^4 + \ldots \\ &= (x+x^2+x^3+x^4+\ldots) + (x^2+x^3+x^4+\ldots) + (x^3+x^4+\ldots) + \ldots,\end{aligned}$$

because if $b_n \geq 2$ then the monomial x^{n+1} is missing at least one times in the above sum. $\square$



The self-counting universe.
The blue point in the middle is the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, \ldots\}$ with all trajectories converging to it.

5. The Extended Self-Counting Flow Φ^* on $\mathbb{Z}^{\mathbb{N}}$

In this section, we extend the self-counting flow Φ from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{Z}^{\mathbb{N}}$, giving us the extended self-counting flow Φ^*.

5.1. **The Extended Self-Counting Flow Φ^*.** We make the following definition.

Definition 5.1 (The extended self-counting flow Φ^*)**.**
We define the extended self-counting flow Φ^* on $\mathbb{Z}^{\mathbb{N}}$ by

$$\Phi^* : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$$

$$\{b_k\}_{k=1}^{\infty} \mapsto \{c_k\}_{k=1}^{\infty} \;\; \textit{if} \; \sum_{k=1}^{\infty}{}^{*} \frac{x^k}{1 - x^{b_k}} = \sum_{k=1}^{\infty} c_k x^k,$$

where the notation $\sum_{k=1}^{\infty}{}^{*}$ means that the term $\frac{x^k}{1-x^{b_k}}$ in the sum is omitted if $b_k = 0$ for some $k \in \mathbb{N}$.

Theorem 5.2 (Explicit formula for $\{c_k\}_{k=1}^{\infty} = \Phi^*\left(\{b_k\}_{k=1}^{\infty}\right)$)**.**
We have for $\{c_k\}_{k=1}^{\infty} = \Phi^*\left(\{b_k\}_{k=1}^{\infty}\right)$ the explicit formula

$$c_k = \delta_{\operatorname{sign}(b_k),1} + \sum_{n=1}^{k-1} \operatorname{sign}(b_n) \mathbb{I}_{b_n|(k-n)}, \tag{5.1}$$

if a sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{Z}^{\mathbb{N}}$ is given.

Proof. We can calculate that

$$\begin{aligned}\sum_{k=1}^{\infty} c_k x^k &= \sum_{k=1}^{\infty}{}^{*} \frac{x^k}{1-x^{b_k}} \\ &= \sum_{n=1}^{\infty}{}^{*} \frac{x^n}{1-x^{b_n}} \\ &= \sum_{n=1}^{\infty} \delta_{\mathrm{sign}(b_n),1} x^n \sum_{m=0}^{\infty} x^{m\cdot b_n} - \sum_{n=1}^{\infty} \delta_{\mathrm{sign}(b_n),-1} x^n \sum_{m=1}^{\infty} x^{m\cdot|b_n|} \\ &= \sum_{k=1}^{\infty} \left(\delta_{\mathrm{sign}(b_k),1} + \sum_{n=1}^{k-1} \mathrm{sign}(b_n) \mathbb{I}_{b_n|(k-n)} \right) x^k.\end{aligned}$$

Equating coefficients on both sides, we obtain the claimed identity (5.1). In the third step of the above calculation, we have used that

$$\frac{x^d}{1-x^{b_d}} = -x^d \frac{x^{|b_d|}}{1-x^{|b_d|}} = -x^d \sum_{m=1}^{\infty} x^{m\cdot|b_d|}, \quad \text{if } b_d < 0.$$

□

5.2. **The Boundary of the Extended Self-Counting Flow Φ^* on $\mathbb{Z}^{\mathbb{N}}$.** We make the following definition.

Definition 5.3 (The boundary $\mathcal{B}_{\Phi^*}$ of the extended self-counting flow Φ^*)**.**
We define the boundary $\mathcal{B}_{\Phi^*}$ of the extended self-counting flow Φ^* on $\mathbb{Z}^{\mathbb{N}}$ by

$$\mathcal{B}_{\Phi^*} := \mathbb{Z}^{\mathbb{N}} \setminus \Phi^*(\mathbb{Z}^{\mathbb{N}}).$$

Theorem 5.4 (The first special non-boundary point)**.**
We have that $\{1,1,1,1,\ldots\} \notin \mathcal{B}_{\Phi^*}$.

Proof. It holds that

$$\Phi^*(\{1,0,0,0,\ldots\}) = \{1,1,1,1,\ldots\} \in \mathcal{B}_{\Phi},$$

because with $\{b_k\}_{k=1}^{\infty} = \{1,0,0,0,\ldots\}$, we have that

$$\sum_{k=1}^{\infty}{}^{*} \frac{x^k}{1-x^{b_k}} = \frac{x}{1-x} = x + x^2 + x^3 + x^4 + \ldots.$$

Therefore, it holds that $\{1,1,1,1,\ldots\} \notin \mathcal{B}_{\Phi^*}$. □

Theorem 5.5 (The second special non-boundary point)**.**
We have that $\{1,0,0,0,0,\ldots\} \notin \mathcal{B}_{\Phi^*}$.

Proof. It holds that

$$\frac{x}{1-x^2} + \frac{x^2}{1-x^{-1}} + \frac{x^4}{1-x^4} + \frac{x^6}{1-x^4} = x$$

and therefore, we have that

$$\Phi^*(\{2,-1,0,4,0,4,0,0,0,0,\ldots\}) = \{1,0,0,0,0,\ldots\}.$$

Therefore $\{1,0,0,0,\ldots\}$ is not a boundary point. □

Theorem 5.6 (The boundary $\mathcal{B}_{\Phi^*}$ is non-empty)**.**
We let M^* be the set

$$\begin{aligned} M^* :&= \prod_{k=1}^{\infty} \{-k+1, \dots, k\} \\ &= \{0,1\} \times \{-1,0,1,2\} \times \{-2,-1,0,1,2,3\} \times \dots . \end{aligned}$$

Then we have that

$$\mathbb{Z}^{\mathbb{N}} \setminus M^* \subset \mathcal{B}_{\Phi^*}.$$

Proof. This follows from $\Phi^*(\mathbb{Z}^{\mathbb{N}}) \subset M^*$, which holds because we get from the explicit formula (5.1) that

$$-k+1 \le c_k = \delta_{\operatorname{sign}(b_k),1} + \sum_{n=1}^{k-1} \operatorname{sign}(b_n) \mathbb{I}_{b_n|(k-n)} \le k$$

for all $k \in \mathbb{N}$. □

5.3. **The Fixed Points of the Extended Self-Counting Flow Φ^* on $\mathbb{Z}^{\mathbb{N}}$.** All fixed points of the extended self-counting flow Φ^* are given by the following theorem.

Theorem 5.7 (The fixed point set of the extended self-counting flow Φ^*)**.**
We have that all fixed points p_n for $n \in \mathbb{N}_0 \cup \{\infty\}$ of the extended self-counting flow Φ^* are given by

$$p_n = \{0\}^n \times \{a_k\}_{k=1}^{\infty} = \{\underbrace{0,0,\dots,0}_{n-times},1,2,2,3,3,3,4,4,4,4,\dots\} \ \textit{for}\ n \in \mathbb{N}_0$$

and

$$p_\infty = \{0,0,0,0,\dots\}.$$

Proof. The sequence $\{b_k\}_{k=1}^{\infty} \in \mathbb{Z}^{\mathbb{N}}$ is a fixed point of the extended self-counting flow Φ^* if

$$\sum_{k=1}^{\infty}{}^{*} \frac{x^k}{1-x^{b_k}} = \sum_{k=1}^{\infty} b_k x^k.$$

Because of the identity

$$\frac{x^k}{1-x^{b_k}} = -x^k \frac{x^{|b_k|}}{1-x^{|b_k|}} = -x^k \sum_{m=1}^{\infty} x^{m \cdot |b_k|} = -x^{k+|b_k|} + \dots \quad \text{for } b_k < 0,$$

we must have that $b_1 \ge 0$. The only two possibilities are $b_1 = 0$ and $b_1 = 1 = a_1$.

If we have that $b_1 = 1 = a_1$, then we must have that $\{b_k\}_{k=1}^{\infty} = \{a_k\}_{k=1}^{\infty}$, where $\{a_k\}_{k=1}^{\infty} = \{1,2,2,3,3,3,\dots\}$ is the self-counting sequence. This is because this case reduces to the case where $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ with its unique solution $\{b_k\}_{k=1}^{\infty} = \{a_k\}_{k=1}^{\infty}$.
If we have that $b_1 = 0$, then either $b_2 = 0$ or $b_2 = 1 = a_1$ by the exact same reasons as before.

If $b_1 = 0$ and $b_2 = 1$, then we must have that $\{b_k\}_{k=1}^{\infty} = \{0\} \times \{a_k\}_{k=1}^{\infty}$ by a similar reasoning as before.
If $b_1 = 0$ and $b_2 = 0$, then it must hold that either $b_3 = 0$ or $b_3 = 1$ again by the same reasons as before. And so on.

The special point $\{b_k\}_{k=1}^{\infty} = \{0\}_{k=1}^{\infty} = \{0,0,0,0,\dots\}$ with $b_k = 0$ for all $k \in \mathbb{N}$ is also a fixed point of the extended self-counting flow Φ^*.

Therefore, we have that all fixed points of the extended self-counting flow Φ^* are either of the form $\{b_k\}_{k=1}^\infty = \{0\}_{k=1}^n \times \{a_k\}_{k=1}^\infty$ for some $n \in \mathbb{N}_0$ or $\{b_k\}_{k=1}^\infty = \{0\}_{k=1}^\infty = \{0, 0, 0, 0, \ldots\}$. $\square$

Corollary 5.8 (The special fixed point $\{a_k\}_{k=1}^\infty$ of Φ and Φ^*)**.**
The self-counting sequence $\{a_k\}_{k=1}^\infty = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$ is also a fixed point of Φ^*. It is the only fixed point contained in $\mathbb{N}^\mathbb{N}$ and all other fixed points are contained in ${\mathbb{N}_0}^\mathbb{N}$.

5.4. **The Non-Invertibility of the Extended Self-Counting Flow Φ^* on $\mathbb{Z}^\mathbb{N}$.** The next theorem tells us that the extended self-counting flow Φ^* on $\mathbb{Z}^\mathbb{N}$ is not invertible.

Theorem 5.9 (Non-invertibility of the extended self-counting flow Φ^*)**.**
We have that the map ${\Phi^*}^{-1}$ does not exist.

Proof. We have that

$$\Phi^*(\{2, 2, 0, 0, 0, 0, \ldots\}) = \Phi^*(\{1, 0, 0, 0, 0, \ldots\}) = \{1, 1, 1, 1, \ldots\},$$

because

$$\frac{x}{1 - x^2} + \frac{x^2}{1 - x^2} = \frac{x}{1 - x}.$$

$\square$

6. The Generalized Self-Counting Flows $\Phi_{\{d_k\}_{k=1}^\infty}$

We start with the following definition.

Definition 6.1 (The generalized self-counting flows $\Phi_{\{d_k\}_{k=1}^\infty}$)**.**
We define for a fixed positive integer sequence $\{d_k\}_{k=1}^\infty \in \mathbb{N}^\mathbb{N}$ the generalized self-counting flows $\Phi_{\{d_k\}_{k=1}^\infty}$ by

$$\Phi_{\{d_k\}_{k=1}^\infty} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$$
$$\{b_k\}_{k=1}^\infty \mapsto \{c_k\}_{k=1}^\infty \quad \text{if} \quad \sum_{k=1}^\infty \frac{d_k x^k}{1 - x^{b_k}} = \sum_{k=1}^\infty c_k x^k.$$

It holds that $\Phi = \Phi_{\{1,1,1,1,\ldots\}}$ is the self-counting flow.

Theorem 6.2 (Explicit formula for $\{c_k\}_{k=1}^\infty = \Phi_{\{d_k\}_{k=1}^\infty}(\{b_k\}_{k=1}^\infty)$)**.**
We have for $\{c_k\}_{k=1}^\infty = \Phi_{\{d_k\}_{k=1}^\infty}(\{b_k\}_{k=1}^\infty)$ the explicit formula

$$c_k = \sum_{n=1}^{k} d_n \mathbb{I}_{b_n | (k-n)}, \tag{6.1}$$

if a sequence $\{b_k\}_{k=1}^\infty \in \mathbb{N}^\mathbb{N}$ is given. This formula is also equivalent to the following expression

$$c_k = d_k + \sum_{n=1}^{k-1} d_n \mathbb{I}_{b_n | (k-n)}. \tag{6.2}$$

Proof. We can calculate that

$$\begin{aligned}\sum_{k=1}^{\infty} c_k x^k &= \sum_{k=1}^{\infty} \frac{d_k x^k}{1-x^{b_k}} \\ &= \sum_{k=1}^{\infty} d_k x^k \sum_{m=0}^{\infty} x^{m\cdot b_k} \\ &= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} d_n \mathbb{I}_{b_n|(k-n)} \right) x^k.\end{aligned}$$

Equating coefficients on both sides, we obtain the claimed identity (6.1). The second formula (6.2) follows, because $b_n|0$ for all $n \in \mathbb{N}$. □

We have for example the following fixed point identity related to $\Phi_{\{1,2,1,2,1,2,1,2,\ldots\}}$, namely

$$\sum_{k=1}^{\infty} \frac{x^{2k-1}}{1-x^{a_{1,2}(2k-1)}} + \sum_{k=1}^{\infty} \frac{2x^{2k}}{1-x^{a_{1,2}(2k)}} = \sum_{k=1}^{\infty} a_{1,2}(k) x^k \tag{6.3}$$

with

$$\{a_{1,2}(k)\}_{k=1}^{\infty} = \{1,3,2,3,5,3,5,5,5,6,5,6,7,6,6,8,6,9,6,9,\ldots\}.$$

Moreover, we have the following fixed point identity related to $\Phi_{\{k\}_{k=1}^{\infty}} = \Phi_{\{1,2,3,4,\ldots\}}$, namely

$$\sum_{k=1}^{\infty} \frac{k x^k}{1-x^{a_k(k)}} = \sum_{k=1}^{\infty} a_k(k) x^k \tag{6.4}$$

with

$$\{a_k(k)\}_{k=1}^{\infty}=\{1,3,4,5,8,7,11,11,14,11,17,13,25,21,19,17,20,26,35,29,\ldots\}.$$

As an application of Theorem 6.2, we have the following two theorems.

Theorem 6.3 (An explicit formula for $a_{1,2}(k)$)**.**
We have for the positive integer sequence $a_{1,2}(k)$, defined in the above equation (6.3), the formula

$$a_{1,2}(k) = \sum_{n=1}^{k} \left(1+\mathbb{I}_{2|n}\right) \mathbb{I}_{a_{1,2}(n)|(k-n)}. \tag{6.5}$$

This formula is also equivalent to the following explicit expression

$$a_{1,2}(k) = \left(1+\mathbb{I}_{2|k}\right) + \sum_{n=1}^{k-1} \left(1+\mathbb{I}_{2|n}\right) \mathbb{I}_{a_{1,2}(n)|(k-n)}. \tag{6.6}$$

Proof. These two formulas follow by applying Theorem 6.2 to the equation (6.3). □

Theorem 6.4 (An explicit formula for $a_k(k)$)**.**
We have for the positive integer sequence $\{a_k(k)\}_{k=1}^{\infty}$, defined in the above equation (6.4), the formula

$$a_k(k) = \sum_{n=1}^{k} n \mathbb{I}_{a_n(n)|(k-n)}. \tag{6.7}$$

This formula is also equivalent to the explicit expression

$$a_k(k) = k + \sum_{n=1}^{k-1} n\mathbb{I}_{a_n(n)|(k-n)}. \tag{6.8}$$

Proof. These two formulas follow by applying Theorem 6.2 to the equation (6.4). □

7. The Generalized Self-Counting Identities

In this section, we give infinitely many generalizations of the self-counting identity (2.1). These generalizations are related to the generalized self-counting flows $\Phi_{\{n,n,n,n,\ldots\}}$ for $n \in \mathbb{N}$ defined in Definition 6.1.

Definition 7.1 (The generalized self-counting sequences)**.**
We define for every natural number $n \in \mathbb{N}$ the generalized self-counting sequence $\{a_k(n)\}_{k=1}^{\infty}$ by

$$\{a_k(n)\}_{k=1}^{\infty} := \{\underbrace{n, n, \ldots, n}_{n-times}, \underbrace{2n, 2n, \ldots, 2n}_{2n-times}, \underbrace{3n, 3n, \ldots, 3n}_{3n-times}, \ldots\}.$$

In particular, we have that $\{a_k\}_{k=1}^{\infty} = \{a_k(1)\}_{k=1}^{\infty}$ is the self-counting sequence.

The generalized self-counting sequence $\{a_k(n)\}_{k=1}^{\infty}$ satisfies the generating function identity

$$\sum_{k=1}^{\infty} a_k(n)x^k = \frac{nx}{1-x}\sum_{k=0}^{\infty} x^{\frac{k^2+k}{2}n} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1. \tag{7.1}$$

This identity holds, because we have that

$$\begin{aligned}\sum_{k=1}^{\infty} a_k(n)x^k &= \frac{nx}{1-x}\left(1 + x^n + x^{3n} + x^{6n} + \ldots\right) \\ &= \frac{nx}{1-x}\sum_{k=0}^{\infty} x^{\frac{k^2+k}{2}n},\end{aligned}$$

which follows from

$$\begin{aligned}&n \cdot \{\underbrace{1, 1, \ldots, 1}_{n-times}, \underbrace{2, 2, \ldots, 2}_{2n-times}, \underbrace{3, 3, \ldots, 3}_{3n-times}, \ldots\} \\ &= \{\underbrace{n, n, \ldots, n}_{n-times}, \underbrace{2n, 2n, \ldots, 2n}_{2n-times}, \underbrace{3n, 3n, \ldots, 3n}_{3n-times}, \ldots\}.\end{aligned}$$

Theorem 7.2 (The generalized self-counting identities)**.**
For every natural number $n \in \mathbb{N}$, we have that

$$\sum_{k=1}^{\infty} \frac{nx^k}{1-x^{a_k(n)}} = \sum_{k=1}^{\infty} a_k(n)x^k \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1, \tag{7.2}$$

where $\{a_k(n)\}_{k=1}^{\infty}$ is a generalized self-counting sequence.

Proof. We can calculate that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{nx^k}{1-x^{a_k(n)}} &= n \cdot \left(\frac{x}{1-x^{a_1(n)}} + \frac{x^2}{1-x^{a_2(n)}} + \frac{x^3}{1-x^{a_3(n)}} + \frac{x^4}{1-x^{a_4(n)}} + \dots \right) \\
&= n \cdot \left(\frac{x}{1-x^n}(1+x+x^2+\dots+x^{n-1}) \right. \\
&\qquad + \frac{x^{n+1}}{1-x^{2n}}(1+x+x^2+\dots+x^{2n-1}) \\
&\qquad + \frac{x^{3n+1}}{1-x^{3n}}(1+x+x^2+\dots+x^{3n-1}) \\
&\qquad \left. + \frac{x^{6n+1}}{1-x^{4n}}(1+x+x^2+\dots+x^{4n-1}) + \dots \right) \\
&= nx \cdot \left(\frac{1}{1-x^n}(1+x+\dots+x^{n-1}) + \frac{x^n}{1-x^{2n}}(1+x+\dots+x^{2n-1}) \right. \\
&\qquad \left. + \frac{x^{3n}}{1-x^{3n}}(1+x+\dots+x^{3n-1}) + \frac{x^{6n}}{1-x^{4n}}(1+x+\dots+x^{4n-1}) + \dots \right) \\
&= nx \sum_{k=0}^{\infty} \frac{x^{\frac{k^2+k}{2}n}}{1-x^{nk+n}} \sum_{m=0}^{nk+n-1} x^m \\
&= nx \sum_{k=0}^{\infty} \frac{x^{\frac{k^2+k}{2}n}}{1-x^{nk+n}} \frac{1-x^{nk+n}}{1-x} \\
&= \frac{nx}{1-x} \sum_{k=0}^{\infty} x^{\frac{k^2+k}{2}n} \\
&= \sum_{k=1}^{\infty} a_k(n) x^k.
\end{aligned}$$

This proves the generalized self-counting identities (7.2). □

For $n = 2$, we get from Theorem 7.2 that we have

$$\sum_{k=1}^{\infty} \frac{2x^k}{1-x^{a_k(2)}} = \sum_{k=1}^{\infty} a_k(2) x^k$$

with $\{a_k(2)\}_{k=1}^{\infty} = \{2, 2, 4, 4, 4, 4, 6, 6, 6, 6, 6, 6, \dots\}$.

For $n = 3$, we get from Theorem 7.2 that

$$\sum_{k=1}^{\infty} \frac{3x^k}{1-x^{a_k(3)}} = \sum_{k=1}^{\infty} a_k(3) x^k$$

with $\{a_k(3)\}_{k=1}^{\infty} = \{3, 3, 3, 6, 6, 6, 6, 6, 6, 9, 9, 9, 9, 9, 9, 9, 9, 9, \dots\}$.

The generalized self-counting identities (7.2) are the fixed point identities of the generalized self-counting flows $\Phi_{\{n,n,n,n,\dots\}}$ with $n \in \mathbb{N}$.

Corollary 7.3 (The generalized self-counting sequence divisor identity)**.**

We have for the generalized self-counting sequence $\{a_k(n)\}_{k=1}^{\infty}$ *the identity*

$$a_k(n) = n \sum_{d=1}^{k} \mathbb{I}_{a_d(n)|(k-d)}. \tag{7.3}$$

This formula is also equivalent to the explicit expression

$$a_k(n) = n + n \sum_{d=1}^{k-1} \mathbb{I}_{a_d(n)|(k-d)}. \tag{7.4}$$

Proof. We can calculate that

$$\begin{aligned}
\sum_{k=1}^{\infty} a_k(n)x^k &= \sum_{k=1}^{\infty} \frac{nx^k}{1-x^{a_k(n)}} \\
&= \sum_{d=1}^{\infty} \frac{nx^d}{1-x^{a_d(n)}} \\
&= \sum_{d=1}^{\infty} nx^d \sum_{m=0}^{\infty} x^{m\cdot a_d(n)} \\
&= \sum_{k=1}^{\infty} \left(\sum_{d=1}^{k} n\mathbb{I}_{a_d(n)|(k-d)} \right) x^k \\
&= \sum_{k=1}^{\infty} \left(n \sum_{d=1}^{k} \mathbb{I}_{a_d(n)|(k-d)} \right) x^k.
\end{aligned}$$

Equating coefficients on both sides, we obtain the claimed formula (7.3). The second formula (7.4) follows, because $a_k(n)|0$ for all $k \in \mathbb{N}$ and all $n \in \mathbb{N}$. □

8. Other Number Flows

A number flow on $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$ transforms a sequence $\{b_k\}_{k=1}^{\infty}$ to a sequence $\{c_k\}_{k=1}^{\infty}$. We have for example the following six interesting number flows, which are very similar to the self-counting flow Φ:

(1) The Fibonacci flow $\mathcal{F}$ on the sequences $\{b_k\}_{k=1}^{\infty} \in \{1\}^2 \times \mathbb{N}^{\mathbb{N}}$ given by

$$\begin{aligned}
\mathcal{F} : \{1\}^2 \times \mathbb{N}^{\mathbb{N}} &\to \{1\}^2 \times \mathbb{N}^{\mathbb{N}} \\
\{b_k\}_{k=1}^{\infty} &\mapsto \{c_k\}_{k=1}^{\infty} \\
\text{if } \sum_{k=1}^{n} b_k &= c_{n+2} - 1 \quad \text{with } b_1 = c_1 = 1 \text{ and } b_2 = c_2 = 1.
\end{aligned}$$

Properties:
- Unique fixed point $\{b_k\}_{k=1}^{\infty} = \{F_k\}_{k=1}^{\infty} = \{1, 1, 2, 3, 5, 8, \ldots\}$ is the Fibonacci sequence
- attractive fixed point
- every trajectory goes to this fixed point
- no cyclic points.

(2) The Golden flow ϕ on the sequences $\{b_k\}_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ defined by

$$\begin{aligned} &\phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}} \\ &\{b_k\}_{k=1}^{\infty} \mapsto \{c_k\}_{k=1}^{\infty} \text{ if } b_k^2 - 1 = c_k \text{ for all } k \in \mathbb{N}. \end{aligned}$$

Properties:
- Fixed points satisfy: $b_k^2 - b_k - 1 = 0 \iff b_k = \frac{1+\sqrt{5}}{2}$ or $b_k = \frac{1-\sqrt{5}}{2}$ for all $k \in \mathbb{N}$
- Cyclic points: for example $\{b_k\}_{k=1}^{\infty} = \{0, 0, 0, 0, \ldots\}$ and $\{b_k\}_{k=1}^{\infty} = \{-1, -1, -1, -1, \ldots\}$.

(3) The ψ-flow on the sequences $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ defined by

$$\begin{aligned} &\psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \\ &\{b_k\}_{k=1}^{\infty} \mapsto \{c_k\}_{k=1}^{\infty} \text{ if } \sum_{k=1}^{\infty} \frac{x^{b_k}}{1 - x^{b_k}} = \sum_{k=1}^{\infty} c_k x^k. \end{aligned}$$

Property:
- We have the explicit formula $c_k = \sum_{n=1}^{\infty} \mathbb{I}_{b_n | k}$.

(4) The Ψ-flow on the sequences $\{b_k\}_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ given by

$$\begin{aligned} &\Psi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}} \\ &\{b_k\}_{k=1}^{\infty} \mapsto \{c_k\}_{k=1}^{\infty} \text{ if } \sum_{k=1}^{\infty} \frac{b_k x^k}{1 - x^k} = \sum_{k=1}^{\infty} c_k x^k. \end{aligned}$$

Properties:
- We have the explicit formula $c_k = \sum_{d|k} b_d = \sum_{d=1}^{k} b_d \mathbb{I}_{d|k}$, which follows from Theorem 6.2
- Unique fixed point: $\{b_k\}_{k=1}^{\infty} = \{0, 0, 0, 0, \ldots\}$
- This fixed point is non-attractive.

(5) The δ-flow on the sequences $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ defined by

$$\begin{aligned} &\delta : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \\ &\{b_k\}_{k=1}^{\infty} \mapsto \{c_k\}_{k=1}^{\infty} \text{ if } c_k = \sum_{n=1}^{k} \mathbb{I}_{b_n | k} \text{ for all } k \in \mathbb{N}. \end{aligned}$$

(6) The τ-flow on the sequences $\{b_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ defined by

$$\begin{aligned} &\tau : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \\ &\{b_k\}_{k=1}^{\infty} \mapsto \{c_k\}_{k=1}^{\infty} \text{ if } \sum_{k=1}^{\infty} \frac{b_k x^k}{1 - x^{b_k}} = \sum_{k=1}^{\infty} c_k x^k. \end{aligned}$$

Property:
- We have the explicit formula $c_k = \sum_{n=1}^{k} b_n \mathbb{I}_{b_n | (k-n)}$, which follows from Theorem 6.2.

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