ON THE GENERALIZED TRIBONACCI ZETA FUNCTION

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Abstract. We obtain a meromorphic continuation of the generalized Tribonacci zeta function to the whole complex plane. The residues of the generalized Tribonacci zeta functions associated to the third-order Jacobsthal, Tribonacci and Narayana sequence at negative integer poles are computed.

1. Introduction

Zeta functions associated to integer sequences are important analytic objects; the most notable example of such a zeta function is the Riemann zeta function defined for complex numbers s with $\mathrm{Re}(s) > 1$ as the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and which can be viewed as the zeta function associated to positive integers.

Zeta functions encode arithmetic properties of sequences they are associated to, and they are related to many mathematical and physical phenomena. Hence, it is of interest to study zeta functions associated to different types of integer sequences.

In 1995 Silverman proposed a problem of finding meromorphic continuation of a zeta function associated to a recurrence sequence of the second order, which was solved independently by Bradley and Darling in 1999 (see [17]). Since then, several authors investigated analytic continuation of classical and multiple zeta functions associated to the second order number sequences. In those works, the corresponding zeta functions are named by the sequence they are associated to.

Meromorphic continuation of the Fibonacci zeta function was given independently by Egami [6] and Navas [14]. In 2013 Kamano [12] obtained meromorphic continuation of the Lucas zeta function with a complete list of poles and their corresponding residues. Kamano also showed that the Lucas zeta function associated to certain special Lucas sequences possesses trivial zeros at even negative integers, and computed its values at negative integers at which it is holomorphic. The Hurwitz-type zeta function associated to the Lucas sequence was studied in [18], where the polar structure of this function was fully described.

Meher and Rout studied meromorphic continuation of the multiple Fibonacci zeta functions [16] and the multiple Lucas zeta functions [13]. A deep study of multiple zeta functions associated with linear recurrence sequences was conducted by Essouabri, Matsumoto and Tsumura in [8]. Elsner, Klyve and Toe [7] deduced meromorphic continuation of the juggling zeta function to the entire complex plane; Behera, Dutta, and Ray [2] meromorphically continued the Lucas-balancing zeta function to the entire complex plane. Sourmelidis in [19] derived properties of the zeta function and the Hurwitz-type zeta function associated to the Beatty sequence. The Lipschitz-Lerch zeta function associated with the Beatty sequence was studied in [1].

In the present paper we introduce the generalized Tribonacci zeta function, which is a zeta function associated to the generalized Tribonacci sequence. We prove in Theorem 3.1 and Corollary 3.2 below that the generalized Tribonacci zeta function possesses a meromorphic continuation to the whole complex plane and identify its poles. We also study its special values at negative integers and investigate the polar structure at negative integers of zeta functions associated to the third-order Jacobsthal sequence, Tribonacci sequence and Narayana sequence. The following Corollary, which is proved in Example 4.5, illustrates results of the paper. We refer to sections 2 and 3 for the missing notation.

Corollary 1.1. *Assume that the argument of the complex conjugate root of the characteristic equation* (4.2) *of the Narayana sequence is not a rational multiple of* π*. Then, the Narayana zeta function* $\zeta_N(s)$ *is holomorphic at negative integers, except at the simple poles* $s=-3\ell$, $\ell\in\mathbb{N}$ *at which its residue equals*

$$\operatorname*{Res}_{s=-3\ell}\zeta_N(s)=\frac{1}{\log\alpha}\binom{3\ell}{2\ell}\binom{2\ell}{\ell}(31)^{-\ell},$$

where α *is the real root of the characteristic equation of the Narayana sequence.*

2. Binet's formula for the generalized Tribonacci numbers

The generalized Tribonacci sequence $\{\mathcal{T}_n(r,s,t)\}_{n\geq 0}$ for fixed nonzero constants r, s and t is defined by the third-order recurrence relation

$$\mathcal{T}_{n+3}=r\mathcal{T}_{n+2}+s\mathcal{T}_{n+1}+t\mathcal{T}_n,\quad n\geq 0, \tag{2.1}$$

where the sequence starts with arbitrary initial values $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$.

The characteristic equation for the recurrence (2.1) is given by

$$x^3-rx^2-sx-t=0. \tag{2.2}$$

The Cardano formula yields that the roots α, β and γ of (2.2) are given by

$$\alpha=\frac{r}{3}+P+Q,\quad \beta=\frac{r}{3}+\omega P+\omega^2Q,\quad \gamma=\frac{r}{3}+\omega^2P+\omega Q,$$

where $\omega=\exp(2\pi i/3)$,

$$P=P(r,s,t)=\left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}+\sqrt{\Delta}\right)^{\frac{1}{3}},\ Q=Q(r,s,t)=\left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}-\sqrt{\Delta}\right)^{\frac{1}{3}}$$

and

$$\Delta=\Delta(r,s,t)=\frac{r^3t}{27}-\frac{r^2s^2}{108}+\frac{rst}{6}-\frac{s^3}{27}+\frac{t^2}{4}.$$

By convention, the second and the third roots are taken to be equal to their principal values (i.e. to the values determined by the principal branch of the logarithm). More precisely, for $\Delta>0$, the value $\sqrt{\Delta}$ is positive and numbers P and Q are real.

Remark 2.1. *Let us note here that the quantity* Δ *defined above is a constant multiple of the discriminant* $D=r^2s^2+4s^3-4r^3t-27t^2-18rst$ *of the equation* (2.2)*. More precisely,* $D=-108\Delta$.

Throughout this paper we assume that $\mathcal{T}_0=0$, $\mathcal{T}_1=\mathcal{T}_2=1$, which are the initial values for the Tribonacci, Narayana and the third-order Jacobsthal sequences (defined through appropriate choices of r, s and t in (2.1)). Assuming that $\alpha\neq\beta$, $\beta\neq\gamma$, and $\alpha\neq\gamma$ the Binet formula for the generalized Tribonacci numbers from [4, Theorem 3.2] reads as

$$\mathcal{T}_n=A\alpha^n+B\beta^n+C\gamma^n, \tag{2.3}$$

where

$$A=\frac{1-\beta-\gamma}{(\alpha-\beta)(\alpha-\gamma)},\quad B=\frac{1-\alpha-\gamma}{(\beta-\alpha)(\beta-\gamma)},\quad C=\frac{1-\alpha-\beta}{(\gamma-\alpha)(\gamma-\beta)}. \tag{2.4}$$

In other words, when the roots α, β, γ of the characteristic equation (2.2) are pairwise distinct, the generalized Tribonacci number $\mathcal{T}_n$ can be written as a linear combination of powers α^n, β^n, and γ^n with coefficients A, B, and C given by (2.4).

When the roots are not distinct, they must be real. Using the generalized Binet formula from [10, Section 3.7] (or [9, Lemma 2.2]), one can show that the $n-$th term of the generalized Tribonacci sequence with the initial conditions $\mathcal{T}_0=0$, $\mathcal{T}_1=\mathcal{T}_2=1$ is given by

$$\mathcal{T}_n=\begin{cases}(A_1n+B_1)\,\alpha^n-B_1\beta^n, & \text{if } \alpha \text{ is a double root and } \beta \text{ is a simple root;}\\ \left(A_2n^2+B_2n\right)\alpha^n, & \text{if } \alpha \text{ is a triple root,}\end{cases} \tag{2.5}$$

where $A_1=\frac{1-\alpha-\beta}{\alpha(\alpha-\beta)}$, $B_1=\frac{2\alpha-1}{(\alpha-\beta)^2}$, $A_2=\frac{1-2\alpha}{2\alpha^2}$ and $B_2=\frac{4\alpha-1}{2\alpha^2}$.

Throughout this paper, we impose the following technical assumptions on parameters r, s and t and values Δ, P, Q related to the cubic equation (2.2):

$$1)\ \Delta>0,\qquad 2)\ \frac{r}{3}+P+Q>1,\qquad 3)\ Pr+Qr+3PQ>0. \tag{2.6}$$

Assumptions (2.6) provide sufficient conditions under which the roots of (2.2) are pairwise distinct and have nice properties, as proved in the following lemma.

Lemma 2.2. *Under assumptions* (2.6) *related to the parameters of the cubic equation* (2.2)*, the root α is real and the roots β and γ are complex-conjugate roots of the equation* (2.2) *that satisfy $\alpha>1$ and $|\beta|=|\gamma|<\alpha$.*

Proof. The first condition in (2.6) implies that α is real and β and γ are a pair of complex conjugate roots of the cubic equation (2.2). The second condition implies that $\alpha>1$. A simple calculation shows that $\alpha^2-|\beta|^2=Pr+Qr+3PQ$. Therefore, the third condition implies $|\beta|<\alpha$. □

The roots β and γ are complex-conjugate, hence we can write $\overline{\gamma}=\beta=|\beta|e^{i\phi}$, for some phase ϕ. Now, it is easy to verify that $\overline{C}=B=|B|e^{i\delta}$, for some $\delta\in[0,2\pi)$.

The cubic equation (2.2) which satisfies assumptions (2.6) can be written as:

$$x^3-(\alpha+2\,|\beta|\cos\phi)\,x^2+\left(2\alpha\,|\beta|\cos\phi+|\beta|^2\right)x-\alpha\,|\beta|^2=0. \tag{2.7}$$

Moreover, under assumptions (2.6), it is trivial to see that $\mathcal{T}_n\sim A\alpha^n$, as $n\to\infty$, hence the series $\sum_{n\ge1}\mathcal{T}_n^{-a}$ converges absolutely for every positive number a.

Remark 2.3. *If the characteristic equation* (2.2) *has a double or a triple root that is greater than one (in which case assumptions* (2.6) *are obviously not satisfied), then $|\mathcal{T}_n|$ grows at least as α^n, as $n\to\infty$, hence the series $\sum_{n\ge1}\mathcal{T}_n^{-a}$ also converges absolutely for every positive number a.*

However, the techniques for their meromorphic continuation used in Section 3 below do not apply in this setting, due to the fact that $\mathcal{T}_n$ can no longer be expressed as a linear combination of the $n-th$ powers of roots (the coefficients are either linear or quadratic in n, see equation (2.5)*). Possibly a different method of continuation based on an integral representation can be employed. We leave this question to an interested reader.*

3. Generalized Tribonacci zeta function

The generalized Tribonacci zeta function associated to the generalized Tribonacci sequence $\{\mathcal{T}_n\}_{n\geq 0}$, with initial conditions $\mathcal{T}_0 = 0$, $\mathcal{T}_1 = \mathcal{T}_2 = 1$ is defined for complex numbers s with $\mathrm{Re}(s) > 0$ by the absolutely convergent series

$$\zeta_{\mathcal{T}}(s) := \sum_{n=1}^{\infty} \frac{1}{\mathcal{T}_n^s}.$$

Recall that we assume that assumptions (2.6) hold true, meaning that $\mathcal{T}_n = A\alpha^n + B\beta^n + C\gamma^n$, where α is a real number bigger than one, β and γ are complex conjugates and A, B, C are given by (2.4).

In the following theorem we prove that $\zeta_{\mathcal{T}}(s)$ can be meromorphically continued to the whole complex plane and identify its poles.

Theorem 3.1. *Let $n_0 \in \mathbb{N}$ be the smallest integer such that $\left|\frac{2|B|}{A}\left(\frac{|\beta|}{\alpha}\right)^n\right| < 1$ for all $n \geq n_0$. For $\mathrm{Re}(s) > 0$ the function $\zeta_{\mathcal{T}}(s)$ can be written as*

$$\begin{aligned}\zeta_{\mathcal{T}}(s) = &\sum_{n=1}^{n_0-1} \frac{1}{(A\alpha^n + B\beta^n + C\gamma^n)^s} \\ &+ A^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{|B|}{A}\right)^k \sum_{l=0}^{k}\binom{k}{l} e^{i\delta(2l-k)}\left(\frac{\beta^{k-l}\gamma^l}{\alpha^{s+k}}\right)^{n_0}\frac{\alpha^{s+k}}{\alpha^{s+k}-\beta^{k-l}\gamma^l}.\end{aligned} \tag{3.1}$$

The expression on the right-hand side of (3.1) provides holomorphic continuation of $\zeta_{\mathcal{T}}(s)$ to the whole complex plane except for possible simple poles at

$$s = s_{k,l,n} = -k + k\frac{\log|\beta|}{\log\alpha} + \frac{(k-2l)\phi + 2n\pi}{\log\alpha}i, \quad k, l \in \mathbb{N}_0, l \leq k, n \in \mathbb{Z}. \tag{3.2}$$

Proof. First, we notice that the inequality $|\beta| < \alpha$ implies that $\frac{2|B|}{A}\left(\frac{|\beta|}{\alpha}\right)^n \to 0$ as $n \to \infty$, hence there exists the smallest integer n_0 such that $\left|\frac{2|B|}{A}\left(\frac{|\beta|}{\alpha}\right)^n\right| < 1$ for all $n \geq n_0$.

Our starting point is the identity $B\beta^n + C\gamma^n = 2|B||\beta|^n\cos(n\phi+\delta)$ which follows from equations $\overline{\gamma} = \beta = |\beta|e^{i\phi}$ and $\overline{C} = B = |B|e^{i\delta}$. Using the Binet formula (2.3) we can write $\zeta_{\mathcal{T}}(s)$ for $\mathrm{Re}(s) > 0$ as

$$\begin{aligned}\zeta_{\mathcal{T}}(s) &= \sum_{n=1}^{\infty}\frac{1}{(A\alpha^n + B\beta^n + C\gamma^n)^s} = \sum_{n=1}^{\infty}\frac{1}{(A\alpha^n + 2|B||\beta|^n\cos(n\phi+\delta))^s} \\ &= \sum_{n=1}^{\infty}(A\alpha^n)^{-s}\left(1 + \frac{2|B|}{A}\left(\frac{|\beta|}{\alpha}\right)^n\cos(n\phi+\delta)\right)^{-s}.\end{aligned}$$

For $n \geq n_0$ one has $\left|\frac{2|B|}{A}\left(\frac{|\beta|}{\alpha}\right)^n\cos(n\phi+\delta)\right| < 1$ hence using the Taylor series expansion of $\left(1 + \frac{2|B|}{A}\left(\frac{|\beta|}{\alpha}\right)^n\cos(n\phi+\delta)\right)^{-s}$ we get

$$\begin{aligned}\zeta_{\mathcal{T}}(s) = &\sum_{n=1}^{n_0-1}\frac{1}{(A\alpha^n + B\beta^n + C\gamma^n)^s} \\ &+ \sum_{n=n_0}^{\infty}(A\alpha^n)^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{2|B|}{A}\right)^k(\cos(n\phi+\delta))^k\left(\frac{|\beta|}{\alpha}\right)^{nk}.\end{aligned}$$

The first sum on the right-hand side of the equation above is obviously holomorphic everywhere. Due to the exponential decay of the terms in the second double sum, for $\mathrm{Re}(s) > 0$ we can interchange the order of summation to get

$$\begin{aligned}
&\sum_{n=n_0}^{\infty} (A\alpha^n)^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{2\,|B|}{A}\right)^k (\cos(n\phi+\delta))^k \left(\frac{|\beta|}{\alpha}\right)^{nk} \\
&\quad = A^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{|B|}{A}\right)^k \sum_{n=n_0}^{\infty} |\beta|^{nk} \alpha^{-(s+k)n} \left(e^{i(n\phi+\delta)} + e^{-i(n\phi+\delta)}\right)^k \\
&\quad = A^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{|B|}{A}\right)^k \sum_{n=n_0}^{\infty} |\beta|^{nk} \alpha^{-(s+k)n} \sum_{l=0}^{k} \binom{k}{l} e^{in\phi(k-2l)} e^{i\delta(k-2l)} \\
&\quad = A^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{|B|}{A}\right)^k \sum_{l=0}^{k} \binom{k}{l} e^{i\delta(k-2l)} \left(\frac{|\beta|^{k-l}\,|\beta|^l}{\alpha^{s+k}} e^{i\phi(k-l)} e^{-i\phi l}\right)^{n_0} \cdot \\
&\quad \cdot \sum_{n=0}^{\infty} \left(\frac{|\beta|^{k-l}\,|\beta|^l}{\alpha^{s+k}} e^{i\phi(k-l)} e^{-i\phi l}\right)^n \\
&\quad = A^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{|B|}{A}\right)^k \sum_{l=0}^{k} \binom{k}{l} e^{i\delta(k-2l)} \left(\frac{\beta^{k-l}\gamma^l}{\alpha^{s+k}}\right)^{n_0} \frac{1}{1-\frac{\beta^{k-l}\gamma^l}{\alpha^{s+k}}} \\
&\quad = A^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{|B|}{A}\right)^k \sum_{l=0}^{k} \binom{k}{l} e^{i\delta(k-2l)} \left(\frac{\beta^{k-l}\gamma^l}{\alpha^{s+k}}\right)^{n_0} \frac{\alpha^{s+k}}{\alpha^{s+k}-\beta^{k-l}\gamma^l}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta_{\mathcal{T}}(s) &= \sum_{n=1}^{n_0-1} \frac{1}{(A\alpha^n + B\beta^n + C\gamma^n)^s} \\
&\quad + A^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{|B|}{A}\right)^k \sum_{l=0}^{k} \binom{k}{l} e^{i\delta(k-2l)} \left(\frac{\beta^{k-l}\gamma^l}{\alpha^{s+k}}\right)^{n_0} \frac{\alpha^{s+k}}{\alpha^{s+k}-\beta^{k-l}\gamma^l}.
\end{aligned}$$

The last expression is holomorphic function on $\mathbb{C}$, except for possible poles. Poles are derived from the equation $\alpha^{s+k} - \beta^{k-l}\gamma^l = 0$, which is equivalent to

$$\exp((s+k)\log\alpha) = \exp(k\log|\beta| + i\phi(k-2l)),$$

hence poles are given by (3.2). □

Corollary 3.2. *Let $2|B\beta| < |A\alpha|$. Then, the function $\zeta_{\mathcal{T}}(s)$ can be meromorphically continued to the whole s-plane and expressed as*

$$\zeta_{\mathcal{T}}(s) := A^{-s} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{-s}{k} \binom{k}{l} \left(\frac{C}{A}\right)^l B^{k-l} \frac{\beta^{k-l}\gamma^l}{\alpha^{k+s} - \beta^{k-l}\gamma^l}.$$

This function is holomorphic except for possible simple poles given by (3.2) and the corresponding residues

$$\operatorname*{Res}_{s=s_{k,l,n}} \zeta_{\mathcal{T}}(s) = \frac{1}{\log\alpha} \binom{-s_{k,l,n}}{k} \binom{k}{l} \frac{B^{k-l}C^l}{A^{s_{k,l,n}+k}}.$$

Proof. The assumption $2|B\beta| < |A\alpha|$ implies that $\left|\frac{2|B|}{A}\left(\frac{|\beta|}{\alpha}\right)^n\right| < 1$ for all $n \geq 1$. Taking $n_0 = 1$ in Theorem 3.1 yields that

$$\zeta_{\mathcal{T}}(s) = A^{-s}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\binom{-s}{k}\binom{k}{l}\left(\frac{C}{A}\right)^l B^{k-l}\frac{\beta^{k-l}\gamma^l}{\alpha^{k+s}-\beta^{k-l}\gamma^l}.$$

The possible poles are identified in Theorem 3.1 and the residue at the pole $s = s_{k,l,n}$ is given by

$$\begin{aligned}\operatorname*{Res}_{s=s_{k,l,n}}\zeta_{\mathcal{T}}(s) &= A^{-s_{k,l,n}}\binom{-s_{k,l,n}}{k}\binom{k}{l}\left(\frac{B}{A}\right)^k\left(\frac{C}{B}\right)^l\beta^{k-l}\gamma^l\lim_{s\to s_{k,l,n}}\frac{s-s_{k,l,n}}{\alpha^{s+k}-\beta^{k-l}\gamma^l}\\ &= \binom{-s_{k,l,n}}{k}\binom{k}{l}\frac{B^{k-l}C^l}{A^{s_{k,l,n}+k}}\lim_{s\to s_{k,l,n}}\frac{s-s_{k,l,n}}{\frac{\alpha^{s+k}}{\beta^{k-l}\gamma^l}-1}.\end{aligned}$$

A simple computation, based on the Taylor series expansion of

$$\frac{\alpha^{s+k}}{\beta^{k-l}\gamma^l} = \exp\left(\log\alpha(s-s_{k,l,n})\right)$$

yields that

$$\operatorname*{Res}_{s=s_{k,l,n}}\zeta_{\mathcal{T}}(s) = \frac{1}{\log\alpha}\binom{-s_{k,l,n}}{k}\binom{k}{l}\frac{B^{k-l}C^l}{A^{s_{k,l,n}+k}}.$$

□

Remark 3.3. *The method of proof of the above theorem can be applied to deduce meromorphic continuation (and identify poles) of a zeta function associated to a sequence satisfying a higher degree recurrence relation with constant coefficients, under certain assumptions on the roots of the corresponding characteristic equation. Assuming that the roots of the characteristic equation are all distinct, the generalized Binet formula (see e.g.* [9, Lemma 2.2]*) applies to show that the $n-$th term in the recurrence sequence is expressed as a linear combination of the $n-$th powers of the roots. If the absolute values of roots are e.g. bigger than one, the above method applies to yield a meromorphic continuation of the zeta function associated to this sequence.*

These assumptions can be relaxed with some more work. It is easy to see that the method works well if e.g. the smallest absolute value of the root is less than one; some roots can have absolute value equal to one and so on. For example, the method can be applied in case when the roots of the characteristic equation are pairwise complex conjugates with different absolute values, at least one of which is bigger than one.

As the degree of the equation increases, the expressions necessarily become more complicated, but with some extra work one can identify poles of the zeta function.

4. Special values at negative integers

From Theorem 3.1 it is easy to see that the generalized Tribonacci zeta function possesses the simple pole at $s = s_{0,0,0} = 0$ with the residue equal to $\frac{1}{\log\alpha}$. This means that the residue at $s = 0$ of the generalized Tribonacci zeta function uniquely determines the real root of the corresponding characteristic equation (2.2).

In this section we will study special values of the generalized Tribonacci zeta function at negative integers. First, we will prove the lemma below which gives sufficient conditions on the zeros of the characteristic equation (2.2) under which the generalized Tribonacci zeta function $\zeta_{\mathcal{T}}(s)$ possesses poles at negative integers.

Lemma 4.1. *With the notation as above, under assumption* (2.6)*, the generalized Tribonacci zeta function* $\zeta_{\mathcal{T}}(s)$ *possesses poles at* $s = -m$*, for some* $m \in \mathbb{N}$ *if and only if:*

(i) $|\beta| = 1$ *and either* m *is even or* $\phi = \frac{2n\pi}{2l-m}$*, for some* $n \in \mathbb{Z}$ *and* $l \in \{0, \dots, m\}$*, or*

(ii) $|\beta| < 1$*, and* m *can be written as* $m = q + k$ *for any two positive integers* k, q *which satisfy the following conditions:* $|\beta| = \alpha^{-q/k}$ *and either* k *is even or* $\phi = \frac{2n\pi}{2l-k}$*, for some* $n \in \mathbb{Z}$ *and* $l \in \{0, \dots, k\}$*.*

Before we prove the lemma, let us note here that integers k, q in the part (ii) are not necessarily coprime. In other words, if k_0 is the smallest positive integer such that $|\beta| = \alpha^{-q_0/k_0}$ for some positive integer q_0, then for all $\ell \in \mathbb{N}$ such that $k = \ell k_0$ satisfies the second condition in part (ii), the generalized Tribonacci zeta function $\zeta_{\mathcal{T}}(s)$ possesses poles at $s = -\ell(q_0 + k_0)$.

Proof. We start by observing that $s = s_{k,l,n}$ is not a pole of $\zeta_{\mathcal{T}}(s)$, if $\binom{-s}{k} = 0$. This means that a negative integer $s = s_{k,l,n}$ can be a pole of $\zeta_{\mathcal{T}}(s)$ only if $|\beta| \leq 1$.

If $|\beta| = 1$, one has $s_{k,l,n} = -m$ if and only if $k = m$ and $(m - 2l)\phi = 2n\pi$, for some $n \in \mathbb{Z}$ and $l \in \{0, \dots, m\}$, which proves part (i).

If $|\beta| > 1$, the number $s_{k,l,n}$ can be a negative integer if and only if $k\frac{\log|\beta|}{\log\alpha}$ is a negative integer, i.e. equal to some $-q$ for $q \in \mathbb{N}$ and $(2l-k)\phi = 2n\pi$, for some $n \in \mathbb{Z}$ and $l \in \{0, \dots, k\}$. This proves part (ii). □

Remark 4.2. *Let us further emphasize the following two special cases.*

1. Assume that the real root $\alpha > 1$ *of the characteristic equation* (2.2) *is a Salem number. Then, the complex conjugate roots* β *and* γ *must have modulus* 1*, i.e.* $\beta = \overline{\gamma} = e^{i\phi}$*, for some* $\phi \in (0, \pi)$*. In this case, the generalized Tribonacci zeta function of the corresponding sequence possesses poles at even negative integers* $s_{2\ell,\ell,0} = -2\ell$ *with the residues*

$$\operatorname*{Res}_{s=s_{2\ell,\ell,0}} \zeta_{\mathcal{T}}(s) = \frac{1}{\log\alpha}\binom{2\ell}{\ell}B^\ell C^\ell = \frac{1}{\log\alpha}\binom{2\ell}{\ell}\left(\frac{(1-\alpha)^2 - 2(1-\alpha)\cos\phi + 1}{4\sin^2\phi(\alpha^2 - 2\alpha\cos\phi + 1)}\right)^\ell.$$

2. Assume that the real root α *of the characteristic equation* (2.2) *is a Pisot number less than* 2*, i.e.* α *is a real algebraic integer greater than one which has all its conjugates inside the unit disc* $|z| < 1$*. Then, it must be a unit, hence* $t = 1$*, which combined with* (2.7) *yields that* $\alpha|\beta|^2 = 1$*, meaning that* $|\beta| = \alpha^{-1/2}$*. In this case, the generalized Tribonacci zeta function possesses poles at negative integers* $s = s_{2\ell,\ell,0} = -3\ell$*,* $\ell \in \mathbb{N}$*. Poles at other negative integers might exist only in case that the number* $e^{i\phi} = \beta/|\beta|$ *is some root of unity. We find this outcome highly unlikely, in view of properties of the conjugate Pisot numbers elaborated in* [11] *and* [5]*. However, we were not able to prove that the argument of a conjugate of the third order Pisot number is not a rational multiple of* π*. We find this question very interesting but out of the scope of the present work.*

Example 4.3. We will first consider the third-order Jacobsthal sequence $J = \{J_n\}_{n\geq 0}$ given by the recurrence $J_{n+3} = J_{n+2} + J_{n+1} + 2J_n$, $n \geq 0$, with the initial values $J_0 = 0$, $J_1 = J_2 = 1$. One can easily verify that the coefficients of the corresponding characteristic cubic equation satisfy assumptions (2.6). The roots of the characteristic equation are given by

$$\alpha = 2, \beta = \overline{\gamma} = e^{\frac{2\pi i}{3}} = \frac{-1 + \sqrt{3}i}{2}.$$

These roots satisfy the first condition in Lemma 4.1, i.e. $|\beta| = |\gamma| = 1$. Since $\phi = \frac{2\pi}{3}$, the zeta function

$$\zeta_J(s) := \sum_{n=1}^{\infty} \frac{1}{J_n^s}$$

associated to the third-order Jacobsthal sequence $J = \{J_n\}_{n\geq 1}$ possesses poles at negative integers $s = -k$ if and only if $k = 2l + 3n$, for some $l \in \mathbb{N}_0$, $l \leq k$, $n \in \mathbb{Z}$. It is easy to see that this is not possible only for $k = 1$. Therefore, the zeta function associated to the third-order Jacobsthal sequence has poles at $s = -k$ for $k \in \mathbb{N}_0 \setminus \{1\}$. An elementary calculation yields values of the constants A, B and C given by (2.4):

$$A = \frac{2}{3}, \quad B = \overline{C} = \frac{-3 - 2\sqrt{3}i}{21}.$$

One can easily verify that $B = \overline{C} = \frac{1}{\sqrt{21}} e^{i\delta}$, where $\delta = \arctan \frac{2\sqrt{3}}{3} - \pi$. The inequality $2|B\beta| < |A\alpha|$ is obviously satisfied, hence Corollary 3.2 applies to determine the residues of the third-order Jacobsthal zeta function at the pole $s = -k$ for $k \in \mathbb{N}_0 \setminus \{1\}$. We get

$$\operatorname*{Res}_{s=-k} \zeta_J(s) = \frac{1}{21^{k/2} \log 2} \sum_{\substack{l \in \{0,1,\ldots,k\} \\ (2l-k) \equiv 0 \pmod 3}} \binom{k}{l} e^{i(k-2l)\delta}.$$

In the case when k is an even number ($k = 2m$ for some $m \in \mathbb{N}$), we get

$$\operatorname*{Res}_{s=-2m} \zeta_J(s) = \frac{1}{21^m \log 2} \left(\binom{2m}{m} + 2 \sum_{\substack{l \in \{0,1,\ldots,m-1\} \\ (l-m) \equiv 0 \pmod 3}} \binom{2m}{l} \cos(2(m-l)\delta) \right).$$

Similarly, in the case when k is an odd number ($k = 2m + 1$, $m \in \mathbb{N}$), we get

$$\operatorname*{Res}_{s=-2m-1} \zeta_J(s) = \frac{2}{21^{m+1/2} \log 2} \sum_{\substack{l \in \{0,1,\ldots,m-1\} \\ (2l-2m-1) \equiv 0 \pmod 3}} \binom{2m+1}{l} \cos((2m - 2l + 1)\delta).$$

Example 4.4. Let us investigate the polar structure of the zeta function associated to the Tribonacci sequence $T = \{T_n\}_{n\geq 0}$ given by the third-order recurrence $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, $n \geq 0$, with initial values $T_0 = 0$, $T_1 = T_2 = 1$. It is easy to verify that the coefficients of the corresponding characteristic equation

$$x^3 - x^2 - x - 1 = 0 \tag{4.1}$$

satisfy assumptions (2.6). The roots of the characteristic equation for the given recurrence are

$$\alpha = \frac{1}{3} \left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} \right) \approx 1.83929, \ \beta = \overline{\gamma} = |\beta| e^{\phi i},$$

where

$$\phi = \arctan \left(\frac{\sqrt{3} \left(\sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} \right)}{2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}} \right) + \pi.$$

Number α is the third order Pisot number less than 2, hence $|\beta| = \alpha^{-1/2} < 1$. According to the discussion from Remark 4.2, part 2, the generalized Tribonacci zeta function possesses poles at negative integers $s = s_{2\ell,\ell,0} = -3\ell$, $\ell \in \mathbb{N}$. A simple computation shows that the inequality $2|B\beta| < |A\alpha|$ is satisfied, hence Corollary 3.2 applies to determine the residues of $\zeta_T(s)$.

Assuming further that the phase ϕ of the Pisot conjugate β is not a rational multiple of π, the residue of the Tribonacci zeta function at the pole $s = -3\ell$, $\ell \in \mathbb{N}$ is given by

$$\operatorname*{Res}_{s=-3\ell} \zeta_T(s) = \frac{1}{\log \alpha} \binom{3\ell}{2\ell} \binom{2\ell}{\ell} A^{\ell} |B|^{2\ell} = \frac{1}{\log \alpha} \binom{3\ell}{\ell, \ell, \ell} (ABC)^{\ell}.$$

Here, we denote by $\binom{3\ell}{\ell,\ell,\ell}$ the multinomial coefficient $\frac{(3\ell)!}{\ell!\ell!\ell!}$.

The product ABC can be calculated using (2.4) and Vieta's formulas for the cubic equation (4.1):

$$\begin{aligned} \alpha + \beta + \gamma &= 1, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

Namely, we get

$$ABC = -\frac{\alpha\beta\gamma}{[(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)]^2} = -\frac{1}{D},$$

where D is the discriminant of the equation (4.1). Trivially, $D = -44$, hence assuming that the phase ϕ of the Pisot conjugate β is not a rational multiple of π, the residue of the Tribonacci zeta function at the pole $s = -3\ell$, $\ell \in \mathbb{N}$ equals

$$\operatorname*{Res}_{s=-3\ell} \zeta_T(s) = \frac{1}{\log \alpha} \binom{3\ell}{\ell, \ell, \ell} (44)^{-\ell}.$$

In general, i.e. without further assumptions on the phase ϕ, the poles of the Tribonacci zeta function are given by $s = -\frac{3k}{2}$, for $k \in 2\mathbb{N}_0$ with the residues

$$\operatorname*{Res}_{s=-\frac{3k}{2}} \zeta_T(s) = \frac{1}{\log \alpha} \sum_{(k,\ell,n)\in A_\phi} \binom{3k/2}{k} \binom{k}{\ell} A^{\frac{k}{2}} |B|^k e^{i(k-2\ell)\delta},$$

where δ is the argument of B and $A_\phi \subseteq 2\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{Z}$ is defined by

$$A_\phi = \{(k, \ell, n) : k \in 2\mathbb{N}_0, \ell \in \mathbb{N}_0, n \in \mathbb{Z}, \ell \le k \quad \text{and} \quad (2\ell - k)\phi = 2n\pi\}.$$

Example 4.5. The Narayana sequence $N = \{N_n\}_{n\ge 0}$ is given by the third-order recurrence $N_{n+3} = N_{n+2} + N_n$, $n \ge 0$, with initial values $N_0 = 0$, $N_1 = N_2 = 1$. Like in the previous example, it is easy to verify that the coefficients satisfy assumptions (2.6) and the roots of the characteristic equation

$$x^3 - x^2 - 1 = 0 \tag{4.2}$$

for the Narayana sequence are given by

$$\alpha = \frac{1}{3}\left(1 + \sqrt[3]{\frac{29}{2} + \frac{3}{2}\sqrt{93}} + \sqrt[3]{\frac{29}{2} - \frac{3}{2}\sqrt{93}}\right) \approx 1.46557,\ \beta = \overline{\gamma} = \alpha^{-1/2} e^{\phi i},$$

where

$$\phi = \arctan\left(\frac{\sqrt{3}\left(\sqrt[3]{\frac{29}{2} + \frac{3}{2}\sqrt{93}} - \sqrt[3]{\frac{29}{2} - \frac{3}{2}\sqrt{93}}\right)}{2 - \sqrt[3]{\frac{29}{2} + \frac{3}{2}\sqrt{93}} - \sqrt[3]{\frac{29}{2} - \frac{3}{2}\sqrt{93}}}\right) + \pi.$$

The root α is the fourth smallest Pisot number, the so-called supergolden ratio (or the Narayana's cows constant) - see [15, sequence $A092526$].

Since the real root of the characteristic equation for the Narayana sequence is the third order Pisot number less than 2; the polar structure of the associated Narayana zeta function $\zeta_N(s) := \sum_{n=1}^{\infty} N_n^{-s}$ is very similar to the one of the Tribonacci zeta function. Namely, after a trivial computation which amounts to verification of the inequality $2|B\beta| < |A\alpha|$, we can apply Corollary 3.2 to determine the residues of $\zeta_N(s)$.

Then, under assumption that the argument of β is not a rational multiple of π, we easily deduce that $\zeta_N(s)$ possesses poles at negative integers $s = -3\ell$, $\ell \in \mathbb{N}$ with residues

$$\operatorname*{Res}_{s=-3\ell} \zeta_N(s) = \frac{1}{\log \alpha} \binom{3\ell}{2\ell} \binom{2\ell}{\ell} A^\ell |B|^{2\ell} = \frac{1}{\log \alpha} \binom{3\ell}{\ell, \ell, \ell} (ABC)^\ell,$$

where A, B and C can be calculated using (2.4) and Vieta's formulas for the cubic equation (4.2). More precisely, we again have the equality $ABC = -1/D$, where D is the discriminant of (4.2), equal to -31. Therefore, assuming that the phase ϕ of the Pisot conjugate β is not a rational multiple of π, the residue of the Narayana zeta function at the pole $s = -3\ell$, $\ell \in \mathbb{N}$ equals

$$\operatorname*{Res}_{s=-3\ell} \zeta_N(s) = \frac{1}{\log \alpha} \binom{3\ell}{\ell, \ell, \ell} (31)^{-\ell}.$$

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