SOME PROPERTIES OF THE FIBONACCI-PASCAL TRIANGLE

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Abstract. This present work introduces a new triangular array modified from the Hosoya triangle and Pascal's triangle and explores some properties. The sum of all elements in each row in the form of a Fibonacci number is derived from a recursion of row sum and the definition of the Fibonacci sequence. An entry expression is formulated by a combinatorial approach using the grid walk problem.

1. Introduction

The Fibonacci sequence $\{f_n\}$ is defined by the recursive relation $f_n = f_{n-1} + f_{n-2}$ where $f_0 = f_1 = 1$. The Hosoya triangle, first introduced by Haruo Hosoya [2], was defined as a set containing products of two Fibonacci numbers $H(m, n) = f_{m-n}f_n$ for all $m \geq n \geq 0$:

$$
\begin{array}{lccccccccccccccccc}
R_0: &&&&&&&&& 1 \\
R_1: &&&&&&&& 1 && 1 \\
R_2: &&&&&&& 2 && 1 && 2 \\
R_3: &&&&&& 3 && 2 && 2 && 3 \\
R_4: &&&&& 5 && 3 && 4 && 3 && 5 \\
R_5: &&&& 8 && 5 && 6 && 6 && 5 && 8 \\
R_6: &&& 13 && 8 && 10 && 9 && 10 && 8 && 13 \\
R_7: && 21 && 13 && 16 && 15 && 15 && 16 && 13 && 21 \\
R_8: & 34 && 21 && 26 && 24 && 25 && 24 && 26 && 21 && 34 \\
&&&&&&&&& \vdots
\end{array}
$$

Figure 1: The Hosoya triangle.

Pascal's triangle is a triangular array with 1's on the boundary and each of the remaining is the sum of the nearest two numbers in the row above. Combinatorially, each entry represents a binomial coefficient $\binom{n}{r} = \dfrac{n!}{(n-r)!r!}$:

$$
\begin{array}{lccccccccccccccccc}
R_0: &&&&&&&&& 1 \\
R_1: &&&&&&&& 1 && 1 \\
R_2: &&&&&&& 1 && 2 && 1 \\
R_3: &&&&&& 1 && 3 && 3 && 1 \\
R_4: &&&&& 1 && 4 && 6 && 4 && 1 \\
R_5: &&&& 1 && 5 && 10 && 10 && 5 && 1 \\
R_6: &&& 1 && 6 && 15 && 20 && 15 && 6 && 1 \\
R_7: && 1 && 7 && 21 && 35 && 35 && 21 && 7 && 1 \\
R_8: & 1 && 8 && 28 && 56 && 70 && 56 && 28 && 8 && 1 \\
&&&&&&&&& \vdots
\end{array}
$$

Figure 2: Pascal's triangle.

A generalization of Pascal's triangle can be constructed by replacing each side of the triangle with any sequences a_n and b_n where $a_0 = b_0$. For $m, n \geq 0$, we define

$$P(m,n) = \begin{cases} a_m, & \text{if } n = 0 \\ b_n, & \text{if } m = 0 \\ P(m, n-1) + P(m-1, n), & \text{otherwise.} \end{cases} \tag{1.1}$$

In the original Pascal's triangle where a_n and b_n are sequences of 1, $P(m,n) = \binom{m+n}{n}$.

$$\begin{array}{lccccccccccc}
R_0: & & & & & & P(0,0) & & & & & \\
R_1: & & & & & P(1,0) & & P(0,1) & & & & \\
R_2: & & & & P(2,0) & & P(1,1) & & P(0,2) & & & \\
R_3: & & & P(3,0) & & P(2,1) & & P(1,2) & & P(0,3) & & \\
R_4: & & P(4,0) & & P(3,1) & & P(2,2) & & P(1,3) & & P(0,4) & \\
R_5: & P(5,0) & & P(4,1) & & P(3,2) & & P(2,3) & & P(1,4) & & P(0,5) \\
 & & & & & & \vdots & & & & &
\end{array}$$

Figure 3: The generalized Pascal's triangle with sequences a_n and b_n boundary.

Inspired by the Hosoya triangle and Pascal's triangle, we introduce the Fibonacci-Pascal triangle, which refers to a special case of the generalized Pascal's triangle where the sequences $a_n = b_n = f_n$:

$$\begin{array}{lccccccccccccccccc}
R_0: & & & & & & & & & 1 & & & & & & & & \\
R_1: & & & & & & & & 1 & & 1 & & & & & & & \\
R_2: & & & & & & & 2 & & 2 & & 2 & & & & & & \\
R_3: & & & & & & 3 & & 4 & & 4 & & 3 & & & & & \\
R_4: & & & & & 5 & & 7 & & 8 & & 7 & & 5 & & & & \\
R_5: & & & & 8 & & 12 & & 15 & & 15 & & 12 & & 8 & & & \\
R_6: & & & 13 & & 20 & & 27 & & 30 & & 27 & & 20 & & 13 & & \\
R_7: & & 21 & & 33 & & 47 & & 57 & & 57 & & 47 & & 33 & & 21 & \\
R_8: & 34 & & 54 & & 80 & & 104 & & 114 & & 104 & & 80 & & 54 & & 34 \\
 & & & & & & & & & \vdots & & & & & & & &
\end{array}$$

Figure 4: The Fibonacci-Pascal triangle.

For $m \geq n \geq 0$,

$$F(m,n) = \begin{cases} f_n, & \text{if } m = 0 \\ f_m, & \text{if } n = 0 \\ F(m, n-1) + F(m-1, n), & \text{otherwise.} \end{cases} \tag{1.2}$$

Note that the notation $F(m,n)$ is equivalent to $P(m,n)$ in (1.1) where $a_m = f_m$ and $b_n = f_n$. Equivalently, if we set $F(m,n) = R(m+n, n)$, then for $m \geq n \geq 0$, we defined element n of row m in the Fibonacci-Pascal triangle by

$$R(m,n) = \begin{cases} f_m, & \text{if } n = 0, m \\ R(m-1, n-1) + R(m-1, n), & \text{otherwise.} \end{cases} \tag{1.3}$$

Throughout this paper, we aim to determine the row sum of the Fibonacci-Pascal triangle using definitions of this triangular array and the Fibonacci sequence. The entry expression generated from the idea of transferring values on the boundary of the triangle to each entry is proven by the aid of geometric identities found in the generalized Pascal's triangle. The solution is also provided with the hypergeometric function.

2. Row Sum

Lemma 2.1. *Given* $S_{R_N} = \sum_{i=0}^{N} R(N,i)$ *the sum of all elements in row* R_N *for all* $N \geq 2$*,*

$$S_{R_N} = 2(S_{R_{N-1}} + f_{N-2}). \tag{2.1}$$

Proof. Consider the equation

$$S_{R_N} = \sum_{i=0}^{N} R(N,i) = R(N,0) + R(N,N) + \sum_{i=1}^{N-1} R(N,i). \tag{2.2}$$

By the definition (1.3),

$$\sum_{i=1}^{N-1} R(N,i) = R(N-1,0) + R(N-1,N-1) + 2\sum_{i=1}^{N-2} R(N-1,i). \tag{2.3}$$

Since $R(N,0) = f_N = R(N,N)$,

$$\begin{aligned} R(N,0) &= R(N-1,0) + R(N-2,0) = R(N-1,0) + f_{N-2} \\ R(N,N) &= R(N-1,N-1) + R(N-2,N-2) = R(N-1,N-1) + f_{N-2}. \end{aligned} \tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned} S_{R_N} &= R(N,0) + R(N,N) + \sum_{i=1}^{N-1} R(N,i) \\ &= 2f_{N-2} + 2\left[R(N-1,0) + R(N-1,N-1) + \sum_{i=1}^{N-2} R(N-1,i)\right] \\ &= 2f_{N-2} + 2\sum_{i=0}^{N-1} R(N-1,i) \\ &= 2(S_{R_{N-1}} + f_{N-2}). \end{aligned} \tag{2.5}$$

■

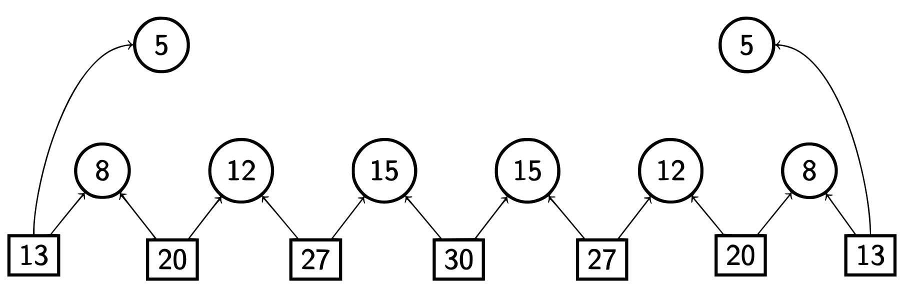

Figure 5: Visual approach to finding S_{R_6}.

Theorem 2.2. *For $N \geq 0$, the sum of all elements in row R_N of the Fibonacci-Pascal triangle is*

$$S_{R_N} = 3 \cdot 2^N - 2f_{N+1}. \tag{2.6}$$

Proof. We use induction to show that

$$S_{R_N} = 2^N + \sum_{i=1}^{N-1} 2^{N-i} f_{i-1}, \tag{2.7}$$

for all $N \geq 2$. Begin with the base case, $S_{R_2} = 6 = 4 + 2(1) = 2^2 + \sum_{i=1}^{2-1} 2^{2-i} f_{i-1}$. Assume this holds for some $N = k \geq 2$. Applying the recursion (2.1), we have

$$S_{R_{k+1}} = 2^{k+1} + \sum_{i=1}^{k-1} 2^{k+1-i} f_{i-1} + 2f_{k-1} = 2^{k+1} + \sum_{i=1}^{k} 2^{k+1-i} f_{i-1}. \tag{2.8}$$

Therefore, by induction, the equation (2.7) holds for all $N \geq 2$. Then consider $X = \sum_{i=1}^{N-1} \frac{f_{i-1}}{2^i}$:

$$\begin{aligned} X &= \frac{f_0}{2} + \frac{f_1}{2^2} + \frac{f_2}{2^3} + \cdots + \frac{f_{N-3}}{2^{N-2}} + \frac{f_{N-2}}{2^{N-1}} \\ 2X &= f_0 + \frac{f_1}{2} + \frac{f_2}{2^2} + \cdots + \frac{f_{N-3}}{2^{N-3}} + \frac{f_{N-2}}{2^{N-2}}. \end{aligned} \tag{2.9}$$

Subtract one from the other and exploit the fact that $f_0 = f_1 = 1$ and $f_N - f_{N-1} = f_{N-2}$, it follows that

$$X = 1 - \frac{f_{N-2}}{2^{N-1}} + \left(\frac{f_0}{2^2} + \frac{f_1}{2^3} + \cdots + \frac{f_{N-4}}{2^{N-2}} \right). \tag{2.10}$$

The sum in the parentheses is equal to $\frac{X}{2} - \frac{f_{N-3}}{2^{N-1}} - \frac{f_{N-2}}{2^N}$. Hence, $X = 2 - \frac{f_{N-1}}{2^{N-2}} - \frac{f_{N-2}}{2^{N-1}}$. Substituting X into (2.7),

$$S_{R_N} = 3 \cdot 2^N - 4f_{N-1} - 2f_{N-2}. \tag{2.11}$$

By the definition of the Fibonacci sequence, the equation above is equal to (2.6). ■

3. Geometric Identities on the Generalized Pascal's Triangle

In Pascal's triangle, the hockey stick theorem states that $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$. Similarly, **the generalized hockey stick identity** appears in the generalized Pascal's triangle:

Theorem 3.1. *For all $m, n \geq 1$,*

$$P(m,n) = P(m,0) + \sum_{i=1}^{n} P(m-1,i) = P(0,n) + \sum_{j=1}^{m} P(j,n-1). \tag{3.1}$$

$$
\begin{array}{lccccccccccc}
R_0: & & & & & & P(0,0) & & & & & \\
R_1: & & & & & P(1,0) & & P(0,1) & & & & \\
R_2: & & & & P(2,0) & & \boxed{\mathrm{P(1,1)}} & & \boxed{\mathrm{P(0,2)}} & & & \\
R_3: & & & P(3,0) & & \boxed{\mathrm{P(2,1)}} & & P(1,2) & & P(0,3) & & \\
R_4: & & P(4,0) & & \boxed{\mathrm{P(3,1)}} & & P(2,2) & & P(1,3) & & P(0,4) & \\
R_5: & P(5,0) & & P(4,1) & & \boxed{\mathrm{P(3,2)}} & & P(2,3) & & P(1,4) & & P(0,5) \\
 & & & & & & \vdots & & & & &
\end{array}
$$

Figure 6: Generalized hockey stick identity for $P(3,2) = P(0,2) + P(1,1) + P(2,1) + P(3,1)$.

This identity can be proven by making use of Pascal's definition $P(m, n-1) + P(m-1, n) = P(m, n)$ for $m, n \geq 1$.

The other identity introduced in this section is called **the hidden Pascal's triangle**. To illustrate the idea, we provide an example of this identity in the Fibonacci-Pascal triangle first. Considering $F(4,3) = 57$, we pick this entry and two more entries in the Fibonacci-Pascal triangle to construct the largest upside-down triangle. In this case, we select $F(4,0) = 5$ and $F(1,3) = 7$ as shown in the figure below:

$$
\begin{array}{lccccccccccccccccc}
R_0: & & & & & & & & & 1 & & & & & & & & \\
R_1: & & & & & & & & 1 & & 1 & & & & & & & \\
R_2: & & & & & & & 2 & & 2 & & 2 & & & & & & \\
R_3: & & & & & & 3 & & 4 & & 4 & & 3 & & & & & \\
R_4: & & & & & \boxed{5} & & \boxed{7} & & \boxed{8} & & \boxed{7} & & 5 & & & & \\
R_5: & & & & 8 & & \boxed{12} & & \boxed{15} & & \boxed{15} & & 12 & & 8 & & & \\
R_6: & & & 13 & & 20 & & \boxed{27} & & \boxed{30} & & 27 & & 20 & & 13 & & \\
R_7: & & 21 & & 33 & & 47 & & \boxed{57} & & 57 & & 47 & & 33 & & 21 & \\
R_8: & 34 & & 54 & & 80 & & 104 & & 114 & & 104 & & 80 & & 54 & & 34 \\
 & & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

Figure 7: The upside-down triangle from $F(4,3) = 57$ on the Fibonacci-Pascal triangle.

Next, we obtain a triangular array from its horizontal reflection and then multiply each entry by a binomial coefficient corresponding to the coordinate of Pascal's triangle. We have all row sums equal to $F(4,3) = 57$ or $F(4,3) = \sum_{i=0}^{h} \binom{h}{i} F(4-i, 3-h+i)$, for all $0 \leq h \leq 3$.

$$
\begin{array}{lccccccc}
h=0: & & & & \binom{0}{0}57 & & & \\
h=1: & & & \binom{1}{0}27 & & \binom{1}{1}30 & & \\
h=2: & & \binom{2}{0}12 & & \binom{2}{1}15 & & \binom{2}{2}15 & \\
h=3: & \binom{3}{0}5 & & \binom{3}{1}7 & & \binom{3}{2}8 & & \binom{3}{3}7
\end{array}
$$

$$\updownarrow$$

$$\begin{array}{lccccccc}
h=0: & & & & \binom{0}{0}(27+30) & & & \\
h=1: & & & \binom{1}{0}(12+15) & & \binom{1}{1}(15+15) & & \\
h=2: & & \binom{2}{0}(5+7) & & \binom{2}{1}(7+8) & & \binom{2}{2}(8+7) & \\
h=3: & \binom{3}{0}5 & & \binom{3}{1}7 & & \binom{3}{2}8 & & \binom{3}{3}7
\end{array}$$

Figure 8: The hidden Pascal's triangle from $F(4,3)$ on the Fibonacci-Pascal triangle.

For any $P(m,n)$ in the generalized Pascal's triangle where $m,n \geq 1$, we obtain a set $V_{P(m,n)}$ consisting of three entries of the generalized Pascal's triangle:

$$V_{P(m,n)} = \begin{cases} \{P(m,n), P(m,0), P(0,n)\}, & \text{if } m = n \\ \{P(m,n), P(m,0), P(m-n,n)\}, & \text{if } m > n \\ \{P(m,n), P(m,n-m), P(0,n)\}, & \text{if } m < n. \end{cases} \tag{3.2}$$

In our example, $V_{F(4,3)} = \{F(4,3), F(4,0), F(1,3)\}$. Construct a new triangular array by forming an upside-down triangle with three entries in $V_{P(m,n)}$ and then, after horizontal reflection of the upside-down triangle, multiplying each entry by a binomial coefficient corresponding to the coordinate of Pascal's triangle. We have all row sums of the new triangular array are equal to $P(m,n)$.

Theorem 3.2. *For all* $0 \leq h \leq \min\{m,n\}$,

$$P(m,n) = \sum_{i=0}^{h} \binom{h}{i} P(m-i, n-h+i). \tag{3.3}$$

This identity follows from Pascal's triangle definition $P(m,n-1) + P(m-1,n) = P(m,n)$ and the binomial identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ for $0 \leq k < n$.

4. Entry Expression

In this section, we attempt to find an expression of each element of the Fibonacci-Pascal triangle through a combinatorial approach.

Given a grid of dimensions $x \times y$. Suppose a particle travels from the top-left corner to the bottom-right corner with each step along an edge of the grid. There are $\binom{x+y}{x} = \binom{x+y}{y}$ shortest paths to reach the destination. In order to complete the journey with the least steps, the particle needs to move to the right x times and down y times in total. Out of $x+y$ steps, we need to choose x steps rightward or y steps downward. The binomial coefficients on Pascal's triangle are able to be superimposed on this walking problem as shown below [4].

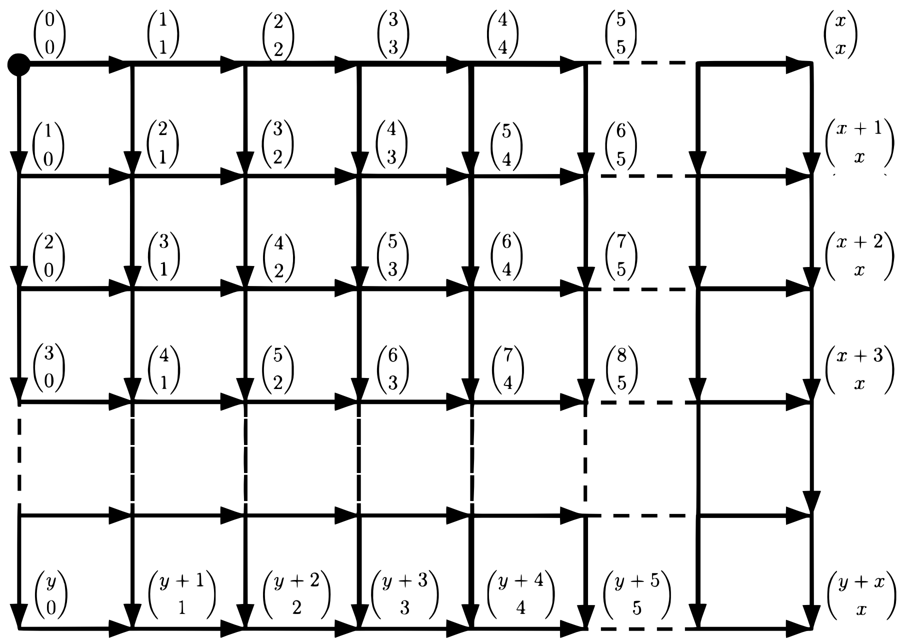

Figure 9: Superimposing Pascal's triangle on the grid walk problem on dimension $x \times y$.

Applying this idea to the generalized Pascal's triangle, each entry can be considered as the value transfers of sequences a_n and b_n from the boundary to that entry on the grid walk problem.

Theorem 4.1. *The entry expression for the generalized Pascal's triangle is of the form*

$$P(m,n) = \sum_{i=1}^{m} \binom{m+n-1-i}{n-1} a_i + \sum_{j=1}^{n} \binom{m+n-1-j}{m-1} b_j, \tag{4.1}$$

for all $m, n \geq 1$.

Proof. We can prove this using induction and the generalized hockey stick identity. Let $S(k)$ be the statement the equation (4.1) holds where either m or n is k and the other is a positive integer less than or equal to k. For the base case, we set $m = n = 1$ to show that $S(1)$ is true: $\binom{0}{0} a_1 + \binom{0}{0} b_1 = P(1,0) + P(0,1) = P(1,1)$. Next, assume the induction hypothesis that for a particular $t \geq 2$, $S(t)$ holds. First, consider $P(t+1, r)$ for any $1 \leq r \leq t$. By **Theorem 3.1**,

$$\begin{aligned} P(t+1,r) &= P(t+1,0) + \sum_{u=1}^{r} P(t,u) \\ &= a_{t+1} + \sum_{u=1}^{r} \left(\sum_{i=1}^{t} \binom{t+u-1-i}{u-1} a_i + \sum_{j=1}^{u} \binom{t+u-1-j}{t-1} b_j \right). \end{aligned} \tag{4.2}$$

Computing each summation with the aid of the hockey stick theorem, we have

$$P(t+1,r) = a_{t+1} + \sum_{i=1}^{t} \binom{t+r-i}{r-1} a_i + \sum_{j=1}^{r} \binom{t+r-j}{t} b_j \tag{4.3}$$
$$= \sum_{i=1}^{t+1} \binom{t+r-i}{r-1} a_i + \sum_{j=1}^{r} \binom{t+r-j}{t} b_j.$$

Likewise, for any $1 \leq r \leq t$,

$$P(r,t+1) = \sum_{i=1}^{r} \binom{t+r-i}{t} a_i + \sum_{j=1}^{t+1} \binom{t+r-j}{r-1} b_j. \tag{4.4}$$

Since $P(t+1,t+1) = P(t+1,t) + P(t,t+1)$, substituting $r = t$ in the two equations above, the sum of these two equations is

$$P(t+1,t+1) = \sum_{i=1}^{t+1} \binom{2t+1-i}{t} a_i + \sum_{j=1}^{t+1} \binom{2t+1-j}{t} b_j. \tag{4.5}$$

Therefore, $S(t+1)$ also holds. By induction, (4.1) is true for all $m,n \geq 1$.

■

Alternatively, the proof of **Theorem 4.1** can be shown by the strong induction (assume $S(2), S(3), ..., S(t)$ hold for a particular $t \geq 2$) and **Theorem 3.2**. Then setting $a_n = b_n = f_n$ in (4.1), we obtain

Corollary 4.2. *The entry expression of the Fibonacci-Pascal triangle is of the form*

$$F(m,n) = \sum_{i=1}^{m} \binom{m+n-1-i}{n-1} f_i + \sum_{j=1}^{n} \binom{m+n-1-j}{m-1} f_j, \tag{4.6}$$

for all $m,n \geq 1$.

Benjamin and Quinn [1] combinatorially proved that the Fibonacci number f_n counts the number of ways to tile a $1 \times n$ board with squares (1×1) and dominoes (1×2). $f_0 = 1$ is the number of tiling an empty space (note that being unable to tile is considered one way). The other different combinatorial problems related to the Fibonacci numbers are also listed on Isaak's note [3]. Altogether, let $W_{x,y}$ be the set containing shortest paths on the grid walk problem of dimension $x \times y$ and T_n be the set containing tiling a $1 \times n$ board with squares and dominoes where $w_{x,y} \in W_{x,y}$ and $t_n \in T_n$. The entry $F(m,n)$ counts all pairs $(w_{n-1,m-i}, t_i)$ and $(w_{m-1,n-j}, t_j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Furthermore, consider the sum

$$\sum_{i=1}^{m} \binom{m+n-1-i}{n-1} z^i = z \frac{(m+n-2)!}{(n-1)!(m-1)!} \sum_{i=1}^{m} \frac{(m+1-i)_{i-1}}{(m+n-i)_{i-1}} z^{i-1}, \tag{4.7}$$

for any $|z| < 1$ and the rising factorial $(c)_k$, for arbitrary $c \in \mathbb{C}$, defined by

$$(c)_k = \begin{cases} 1 & \text{for } k = 0 \\ c(c+1)(c+2)\cdots(c+k-1) & \text{for } k \geq 1. \end{cases} \tag{4.8}$$

Interchanging i with $k = i-1$ in (4.7), we have

$$\sum_{i=1}^{m}\binom{m+n-1-i}{n-1}z^i = z\binom{m+n-2}{n-1}\sum_{k=0}^{m-1}\frac{(m-k)_k}{(m+n-1-k)_k}z^k. \tag{4.9}$$

Because $(c-k)_k = (-1)^k(1-c)_k$, the sum above is equivalent to

$$\sum_{i=1}^{m}\binom{m+n-1-i}{n-1}z^i = z\binom{m+n-2}{n-1}\sum_{k=0}^{m-1}\frac{(1-m)_k}{(2-m-n)_k}z^k. \tag{4.10}$$

Due to the fact that $\dfrac{(1-m)_k}{(2-m-n)_k}z^k = 0$ for all $k \geq m$,

$$\sum_{i=1}^{m}\binom{m+n-1-i}{n-1}z^i = z\binom{m+n-2}{n-1}\sum_{k=0}^{\infty}\frac{(1-m)_k}{(2-m-n)_k}z^k. \tag{4.11}$$

Using $(1)_k = k!$, we obtain

$$\sum_{i=1}^{m}\binom{m+n-1-i}{n-1}z^i = z\binom{m+n-2}{n-1}\sum_{k=0}^{\infty}\frac{(1)_k(1-m)_k}{(2-m-n)_k}\frac{z^k}{k!}. \tag{4.12}$$

By the definition of the hypergeometric function ${}_2F_1(a,b;c;z) = \displaystyle\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$,

$$\sum_{i=1}^{m}\binom{m+n-1-i}{n-1}z^i = z\binom{m+n-2}{n-1}{}_2F_1(1,1-m;2-m-n;z), \tag{4.13}$$

and then

$$\sum_{j=1}^{n}\binom{m+n-1-j}{m-1}z^j = z\binom{m+n-2}{m-1}{}_2F_1(1,1-n;2-m-n;z). \tag{4.14}$$

In equation (4.6), we apply Binet's formula $f_n = F_{n+1} = \dfrac{\varphi^{n+1}-(1-\varphi)^{n+1}}{\sqrt{5}}$ for the golden ratio $\varphi = \dfrac{1+\sqrt{5}}{2}$. It follows that

$$\sum_{i=1}^{m}\binom{m+n-1-i}{n-1}f_i = \frac{\varphi}{\sqrt{5}}\left[\sum_{i=1}^{m}\binom{m+n-1-i}{n-1}\varphi^i - \sum_{i=1}^{m}\binom{m+n-1-i}{n-1}(1-\varphi)^i\right] \tag{4.15}$$

$$\sum_{j=1}^{n}\binom{m+n-1-j}{m-1}f_j = \frac{\varphi}{\sqrt{5}}\left[\sum_{j=1}^{n}\binom{m+n-1-j}{m-1}\varphi^j - \sum_{j=1}^{n}\binom{m+n-1-j}{m-1}(1-\varphi)^j\right].$$

The following corollary is derived from substituting (4.13) and (4.14) into (4.15) and exploiting the fact that $\binom{m+n-2}{n-1} = \binom{m+n-2}{m-1}$:

Corollary 4.3. *For all $m, n \geq 1$,*

$$F(m,n) = \frac{1}{\sqrt{5}}\binom{m+n-2}{m-1}G(m,n), \tag{4.16}$$

where

$$G(m,n) = \varphi^2 {}_2F_1(1, 1-m; 2-m-n; \varphi) - (1-\varphi)^2 {}_2F_1(1, 1-m; 2-m-n; 1-\varphi) \quad (4.17)$$
$$+ \varphi^2 {}_2F_1(1, 1-n; 2-m-n; \varphi) - (1-\varphi)^2 {}_2F_1(1, 1-n; 2-m-n; 1-\varphi).$$

This also yields another result from the row sum formula **Theorem 2.2**:

Corollary 4.4. *For $m, n \geq 1$ and $m + n = N$,*

$$\sum_{m,n} \frac{1}{\sqrt{5}} \binom{m+n-2}{m-1} G(m,n) = S_{R_N} - 2f_N = 3 \cdot 2^N - 2f_{N+2}. \quad (4.18)$$

5. Further Work

Exploring different closed-form solutions for the entry expression of the Fibonacci-Pascal triangle (4.6) would yield new results and alternative interpretations. The further step is to investigate if the generalization of Pascal's triangle (1.1) relates to some generalized binomial expansion. The result from applying the visual approach (Figure 5) to determine the row sum S_{R_N} for the generalized Pascal's triangle is

$$S_{R_N} = 2S_{R_{N-1}} + a_N + b_N - a_{N-1} - b_{N-1}. \quad (5.1)$$

This recursion is derived from Pascal's definition $P(m,n) = P(m, n-1) + P(m-1, n)$ without using the property of Fibonacci numbers. This can begin with the assumption that the sequences a_n and b_n are linear recurrences in general.

The combinatorial approach to obtain the entry expression (4.1) of the generalized Pascal's triangle in the previous section is the concept of how many paths we can transfer each value on the boundary to a particular entry. It would be interesting to see if this idea applies to Pascal-like objects in three-dimensional space or higher dimensions.

Since Pascal's triangle (and its generalization) is an equilateral triangle in two-dimensional space, we define and illustrate the Pascal-like object for the three-dimensional case with a tetrahedron.

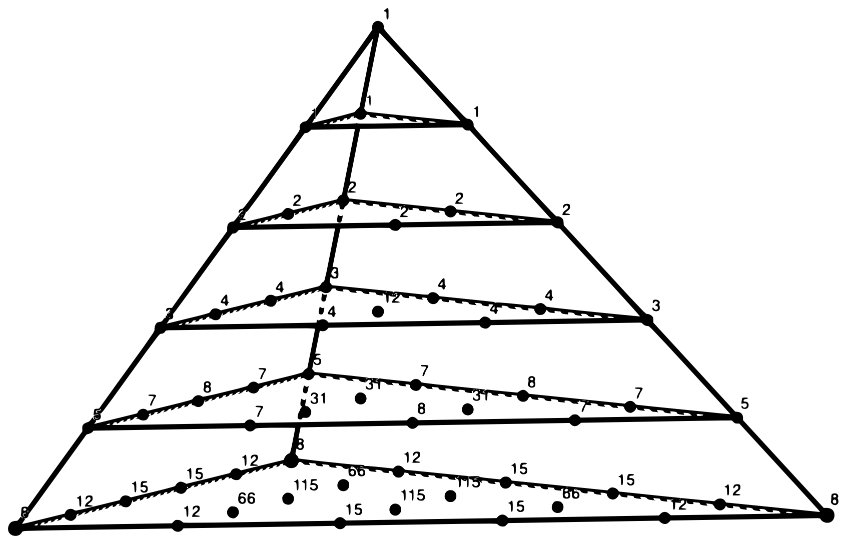

Figure 10: The Fibonacci-Pascal tetrahedron.
`https://www.geogebra.org/m/bdjcjusv`

The figure shown above is a special case related to the Fibonacci-Pascal triangle. The edges of the tetrahedron consist of three Fibonacci sequences, hence the Fibonacci-Pascal triangles on the surfaces. Intersections with horizontal planes at different levels lead to triangular arrays inside the tetrahedron:

Level	Intersection
0	1
1	1 1 1
2	2 2 2 2 2 2
3	3 4 4 4 12 4 3 4 4 3
4	5 7 7 8 31 8 7 31 31 7 5 7 8 7 5
5	8 12 8 15 66 12 15 115 115 15 12 66 115 66 12 8 12 15 15 12 8

Table 1: The intersection of the Fibonacci-Pascal tetrahedron with horizontal planes.

Each side on the boundary of the intersection is the corresponding row of the Fibonacci Pascal's triangle, and each entry inside is the sum of the nearest six numbers in the level above.

Entry	Sum
3 4 4 4 $\boxed{12}$ 4 3 4 4 3	$\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$ $\boxed{2}$
5 7 7 8 31 8 7 31 $\boxed{31}$ 7 5 7 8 7 5	3 4 $\boxed{4}$ 4 $\boxed{12}$ $\boxed{4}$ 3 $\boxed{4}$ $\boxed{4}$ $\boxed{3}$
8 12 8 15 66 12 15 $\boxed{115}$ 115 15 12 66 115 66 12 8 12 15 15 12 8	5 $\boxed{7}$ 7 $\boxed{8}$ $\boxed{31}$ 8 $\boxed{7}$ $\boxed{31}$ $\boxed{31}$ 7 5 7 8 7 5

Table 2: The entry (left) is the sum of the nearest six numbers in the level above (right).

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