ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
ROBERT FRONTCZAK

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany, or by e-mail at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-921 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let r be an even positive integer.

(i) For integers $0 < a < b$ such that $a + b = r$, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+a} F_{n+b} F_{n+r}} \qquad \text{is a rational number.}$$

(ii) For integers $0 < a < b < c < d$ such that $a + d = b + c = r$, prove that

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+a} F_{n+b} F_{n+c} F_{n+d} F_{n+r}} \qquad \text{is a rational number.}$$

H-922 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove that

$$\sum_{n=1}^{\infty} \frac{4^n}{L_{2^n}^2} \sum_{k=1}^{n} \frac{F_{2^k}}{4^k} = \frac{5-\sqrt{5}}{10}.$$

H-923 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania

(i) If $(x_n)_{n\geq 0}$ is a real positive sequence such that $x_0 = 2$, $x_1 = 1$, and

$$\sqrt{F_n^2 + F_{2n}^2} + \sqrt{1 + x_n^2} = \sqrt{(x_n + F_{2n})^2 + (1 + F_n)^2},$$

then find $(x_n)_{n\geq 0}$.

(ii) If $(y_n)_{n\geq 0}$ is a real positive sequence such that $y_0 = 0$, $y_1 = 1$, and

$$\sqrt{L_n^2 + F_{2n}^2} + \sqrt{1 + y_n^2} = \sqrt{(y_n + F_{2n})^2 + (1 + L_n)^2},$$

then find $(y_n)_{n\geq 0}$.

<u>H-924</u> Proposed by Toyesh Prakash Sharma, Agra, India

For $n \geq 0$, prove that

$$\frac{e^{\tan\frac{1}{F_n}} - e^{\sin\frac{1}{F_n}}}{2\tan\frac{1}{F_n} - 2\sin\frac{1}{F_n}} + \frac{e^{\tan\frac{1}{L_n}} - e^{\sin\frac{1}{L_n}}}{2\tan\frac{1}{L_n} - 2\sin\frac{1}{L_n}} \geq e^{\frac{F_{n+1}}{F_{2n}}}.$$

<u>H-925</u> Proposed by the editor

Prove that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n\,16^n} J_{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n\,9^n} J_{2n} = \frac{1}{6}\ln(2),$$

where $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$, $\Re(s) > 1$ is the Riemann zeta function, and J_n are the Jacobsthal numbers given by $J_0 = 0$, $J_1 = 1$, and for all $n \geq 2$, $J_n = J_{n-1} + 2J_{n-2}$.

SOLUTIONS

<u>H-889</u> Proposed by Kapil Kumar Gurjar, Uttar Pradesh, India (Vol. 60, No. 1, February 2022)

Let $\lfloor x \rfloor$ and $\lceil x \rceil$ be the integer part and the ceiling of the real number x, respectively. Then for each integer $m \geq 2$ and $n \geq 2^{m+1} + 1$, we have

(1) $\lfloor \sqrt[2^m]{F_n} \rfloor = \lfloor \sqrt[2^m]{F_{n-2^m}} + \sqrt[2^m]{F_{n-2^{m+1}}} \rfloor$.

(2) The equality (1) above holds with F replaced by L. It also holds with either of F and L and with the floor function replaced by the ceiling function.

(3) $\sqrt[2^m]{F_n} - \sqrt[2^m]{F_{n-2^m}} - \sqrt[2^m]{F_{n-2^{m+1}}} > 1/10^{2^m}$ and the same inequality holds with F replaced by L.

No complete solution was submitted for this problem proposal. The problem remains open. A partial solution was submitted by Dmitry Fleischman.

<u>H-890</u> Proposed by Ryan Zielinski, Clifton, NJ (Vol. 60, No. 1, February 2022)

Prove that for nonnegative integers m,

$$\frac{1}{5^m}\sum_{j=0}^{m} \binom{2m+1}{m-j} L_{2j+1}(2j+1) = 2m+1.$$

Solution by Séan M. Stewart, Thuwal, Saudi Arabia

We will make use of the following result. If n is a nonnegative integer, then

$$\sin^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} \sin((2k+1)x). \qquad (1)$$

A proof is given in the Appendix. Differentiating both sides of this result with respect to x gives

$$(2n+1)\sin^{2n} x \cos x = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} (2k+1)\cos((2k+1)x).$$

Recall

$$T_n(\cos x) = \cos(nx) \qquad \Rightarrow \qquad T_{2k+1}(\cos x) = \cos((2k+1)x),$$

where T_n denotes the Chebyshev polynomials of the first kind. In terms of these polynomials, the above expression may be rewritten as

$$(2n+1)\sin^{2n} x \cos x = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} (2k+1) T_{2k+1}(\cos x).$$

If we let $z = \cos x$, then $\sin x = \pm\sqrt{1-z^2}$ and we see that

$$(2n+1)z(1-z^2)^n = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} (2k+1) T_{2k+1}(z).$$

Setting $z = i/2$ produces

$$(2n+1)\frac{5^n}{2^{2n}}\frac{i}{2} = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} (2k+1) T_{2k+1}\left(\frac{i}{2}\right). \tag{2}$$

Now, it is known that $T_n\left(\frac{i}{2}\right) = \frac{i^n}{2} L_n$, where L_n denotes the nth Lucas number. On shifting the index $n \mapsto 2k+1$, we find

$$T_{2k+1}\left(\frac{i}{2}\right) = \frac{1}{2}(-1)^k i L_{2k+1}.$$

Substituting this into (2), simplifying, and rearranging, we find the required expression.

Appendix: Here we give a proof of (1). Because by definition

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

where i is the imaginary unit, on taking the $(2n+1)$th power of both sides, we get

$$\begin{aligned}
\sin^{2n+1} x &= \frac{(-1)^n}{2^{2n} 2i} \left(e^{ix} - e^{-ix}\right)^{2n+1} \\
&= \frac{(-1)^n}{2^{2n} 2i} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} e^{i(2n-2k+1)x} \\
&= \frac{(-1)^n}{2^{2n} 2i} \left(\sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} e^{i(2n-2k+1)x} + \sum_{k=n+1}^{2n+1} (-1)^k \binom{2n+1}{k} e^{i(2n-2k+1)x} \right).
\end{aligned}$$

We now reindex the sums in the last equation. In the first on the right let $k \mapsto n-k$, while in the second on the right let $k \mapsto k+n+1$. Thus,

$$\sin^{2n+1} x = \frac{1}{2^{2n} 2i} \left(\sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} e^{i(2k+1)x} - \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k+n+1} e^{-i(2k+1)x} \right).$$

By the symmetry rule for the binomial coefficients, namely,

$$\binom{2n+1}{n+k+1} = \binom{2n+1}{2n+1-(n+k+1)} = \binom{2n+1}{n-k},$$

we finally find that

$$\begin{aligned}\sin^{2n+1} x &= \frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}\left(\frac{e^{i(2k+1)x}-e^{-i(2k+1)x}}{2i}\right)\\ &= \frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}\sin((2k+1)x).\end{aligned}$$

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Ángel Plaza, Jason L. Smith, Albert Stadler, and the proposer.

H-891 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 60, No. 1, February 2022)

Prove that for all $n \geq 1$,

$$\frac{F_nF_{10n}-F_{2n}F_{5n}}{5F_n^2} \equiv 0 \pmod{10}.$$

Solution by Won Kyun Jeong, Daegu, South Korea

It is known that $F_{2n} = F_nL_n$, $F_m|F_{mn}$, and

$$L_{m+n} - L_{m-n} = 5F_mF_n \qquad \text{if } n \text{ is even.}$$

We can calculate

$$\begin{aligned}\frac{F_nF_{10n}-F_{2n}F_{5n}}{5F_n^2} &= \frac{F_nF_{5n}L_{5n}-F_nL_nF_{5n}}{5F_n^2}\\ &= \frac{F_{5n}(L_{5n}-L_n)}{5F_n}\\ &= \frac{F_{5n}5F_{3n}F_{2n}}{5F_n}\\ &= F_{5n}F_{3n}L_n.\end{aligned}$$

Because $F_5|F_{5n}, F_3|F_{3n}$, and $F_5 = 5$ and $F_3 = 2$, we get the result.

Also solved by Michel Bataille, Brian Bradie, Charles K. Cook, Nandan Sai Dasireddy, Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher, Jason L. Smith, Albert Stadler, David Terr, Andrés Ventas, and the proposer.

H-892 Proposed by Toyesh Prakash Sharma, Agra, India (Vol. 60, No. 1, February 2022)

Evaluate the following limit

$$\lim_{n\to\infty}\frac{L_{n-1}+L_{n+1}}{n\sqrt[n]{n!}}\sqrt[n+1]{1+\sum_{m=1}^{n}\int_0^1 m^2\ln^{m-1}(1/x)dx\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{F_{n-1}}\right)}.$$

Solution by Brian Bradie, Newport News, VA

By repeated integration by parts,

$$\int_0^1 \ln^{m-1}(1/x)dx = (-1)^{m-1}\int_0^1 \ln^{m-1} x dx = (m-1)!.$$

Thus,

$$1+\sum_{m=1}^{n}\int_0^1 m^2 \ln^{m-1}(1/x)dx = 1+\sum_{m=1}^{n} m\cdot m! = 1+\sum_{m=1}^{n}((m+1)!-m!) = (n+1)!.$$

Now,

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} = 1 \qquad \text{and} \qquad \lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{1}{F_k} = \sum_{k=2}^{\infty}\frac{1}{F_k} < \infty.$$

However,

$$\lim_{n\to\infty}\frac{L_{n-1}+L_{n+1}}{n} \to \infty.$$

Hence,

$$\lim_{n\to\infty}\frac{L_{n-1}+L_{n+1}}{n\sqrt[n]{n!}}\sqrt[n+1]{1+\sum_{m=1}^{n}\int_0^1 m^2\ln^{m-1}(1/x)dx}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{F_{n-1}}\right) \to \infty.$$

Additional remark: If the factor n in the denominator of the first fraction is replaced by α^n, then

$$\lim_{n\to\infty}\frac{L_{n-1}+L_{n+1}}{\alpha^n} = \lim_{n\to\infty}\frac{5F_n}{\alpha^n} = \sqrt{5},$$

and

$$\lim_{n\to\infty}\frac{L_{n-1}+L_{n+1}}{\alpha^n\sqrt[n]{n!}}\sqrt[n+1]{1+\sum_{m=1}^{n}\int_0^1 m^2\ln^{m-1}(1/x)dx}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{F_{n-1}}\right)$$
$$= \sqrt{5}\sum_{k=2}^{\infty}\frac{1}{F_k}.$$

Also solved by Michel Bataille, Dmitry Fleischman, and Albert Stadler.

Editor's remark: The proposer submitted a solution claiming that the limit is 5α.

H-893 Proposed by Séan M. Stewart, Thuwal, Saudi Arabia (Vol. 60, No. 1, February 2022)

Prove

$$\sum_{n=1}^{\infty}\frac{\overline{H}_n F_n}{2^n} = \log\left(\frac{5}{4}\right) + \frac{6}{\sqrt{5}}\log\alpha.$$

Here, $\overline{H}_n$ is the nth skew harmonic number defined by $\sum_{k=1}^{n}(-1)^{k+1}/k$.

Solution by Ángel Plaza, Gran Canaria, Spain

The generating function of the skew harmonic numbers is

$$\sum_{n=1}^{\infty} \overline{H}_n t^n = \frac{\log(1+t)}{1-t} \qquad \text{for} \qquad |t| < 1.$$

So,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\overline{H}_n F_n}{2^n} &= \frac{1}{\sqrt{5}} \left(\frac{\log\left(1+\frac{\alpha}{2}\right)}{1-\frac{\alpha}{2}} - \frac{\log\left(1+\frac{\beta}{2}\right)}{1-\frac{\beta}{2}} \right) \\ &= \log\left(\frac{5}{4}\right) + \frac{6}{\sqrt{5}} \coth^{-1}(\sqrt{5}) \\ &= \log\left(\frac{5}{4}\right) + \frac{6}{\sqrt{5}} \log \alpha,\end{aligned}$$

where it has been used that

$$\coth^{-1}(x) = \frac{1}{2} \log\left(\frac{x+1}{x-1}\right).$$

Also solved by Michel Bataille, Brian Bradie, Nandan Sai Dasireddy, Dmitry Fleischman, Won Kyun Jeong, Raphael Schumacher, Albert Stadler, David Terr, Andrés Ventas, and the proposer.

H-894 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 60, No. 1, February 2022)

For any integers r and $n \geq 1$, prove that

(i) $\displaystyle\sum_{k=1}^{n} F_k F_{k+r} F_{2k+r} = F_n F_{n+1} F_{n+r} F_{n+r+1}$;

(ii) $\displaystyle\sum_{k=1}^{n} F_k^5 F_{k+1}^5 F_{2k+1} = \frac{1}{8}(F_n F_{n+1} F_{n+2})^4$.

Solution by Jason L. Smith, Decatur, IL

Use induction for part (i): First note that if $n = 1$, both sides are equal to $F_{r+1}F_{r+2}$. Now consider the sum with upper limit equal to $n+1$, and use the induction hypothesis

$$\begin{aligned}\sum_{k=1}^{n+1} F_k F_{k+r} F_{2k+r} &= F_{n+1} F_{n+r+1} F_{2n+r+2} + \sum_{k=1}^{n} F_k F_{k+r} F_{2k+r} \\ &= F_{n+1} F_{n+r+1} F_{2n+r+2} + F_n F_{n+1} F_{n+r} F_{n+r+1} \\ &= F_{n+1} F_{n+r+1} \left(F_{2n+r+2} + F_n F_{n+r}\right).\end{aligned}$$

Now, use the identity $F_{p+q} = F_{p+1}F_{q+1} - F_{p-1}F_{q-1}$ [1] to observe that

$$F_{2n+r+2} = F_{n+2} F_{n+r+2} - F_n F_{n+r}.$$

From this, we obtain

$$\sum_{k=1}^{n+1} F_k F_{k+r} F_{2k+r} = F_{n+1} F_{n+r+1} F_{n+2} F_{n+r+2},$$

completing the induction step and confirming the identity.

For part (ii), proving a preliminary result will be useful. Consider the expression $F_{n+3}^4 - F_n^4$, which can be factored as follows

$$F_{n+3}^4 - F_n^4 = (F_{n+3} - F_n)(F_{n+3} + F_n)(F_{n+3}^2 + F_n^2).$$

The first two factors can be simplified using the basic Fibonacci recursion as $2F_{n+1}$ and $2F_{n+2}$, respectively. The third factor can also be rewritten using the basic recursion as

$$F_{n+3}^2 + F_n^2 = (F_{n+2} + F_{n+1})^2 + (F_{n+2} - F_{n+1})^2 = 2(F_{n+2}^2 + F_{n+1}^2).$$

The remaining expression in parentheses can be rewritten using the identity $F_p^2 + F_{p+2q+1}^2 = F_{2q+1}F_{2p+2q+1}$ [1] and we establish that

$$F_{n+3}^4 - F_n^4 = 2F_{n+1}2F_{n+2}2F_{2n+3} = 8F_{n+1}F_{n+2}F_{2n+3}.$$

Now, we use induction for part (ii). Observe that if $n = 1$, both sides are equal to 2. Set the upper limit of the sum to $n + 1$ and proceed with the induction hypothesis using the preliminary result to get

$$\begin{aligned}
\sum_{k=1}^{n+1} F_k^5 F_{k+1}^5 F_{2k+1} &= F_{n+1}^5 F_{n+2}^5 F_{2n+3} + \sum_{k=1}^{n} F_k^5 F_{k+1}^5 F_{2k+1} \\
&= F_{n+1}^5 F_{n+2}^5 F_{2n+3} + \frac{1}{8}(F_n F_{n+1} F_{n+2})^4 \\
&= \frac{1}{8}(F_{n+1}F_{n+2})^4(8F_{n+1}F_{n+2}F_{2n+3} + F_n^4) \\
&= \frac{1}{8}(F_{n+1}F_{n+2})^4(F_{n+3}^4 - F_n^4 + F_n^4) \\
&= \frac{1}{8}(F_{n+1}F_{n+2}F_{n+3})^4,
\end{aligned}$$

which completes the induction step.

Reference

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc., New York, 2001.

Also solved by Michel Bataille, Brian Bradie, Charles K. Cook, Kenny Davenport, Dmitry Fleischman, Won Kyun Jeong, Wei-Kai Lai and John Risher (jointly), Ángel Plaza, Albert Stadler, David Terr, and the proposer.