# ADVANCED PROBLEMS AND SOLUTIONS

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*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany, or by email at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.*

## PROBLEMS PROPOSED IN THIS ISSUE

**H-911 Proposed by Hideyuki Ohtsuka, Saitama, Japan**

Let $r \geq 2$ be an even number, $s$ be an integer, and $\mathbf{i} = \sqrt{-1}$. Prove that

(i) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{F_r}{F_{rn+s}}\right) = \frac{1+\beta^s}{1-\beta^{r+s}}$, if $s \geq 0$ is even;

(ii) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{F_r}{F_{rn+s}}\mathbf{i}\right) = \frac{\alpha^s+\mathbf{i}}{\alpha^s-\beta^r\mathbf{i}}$, if $s$ is odd.

**H-912 Proposed by Hideyuki Ohtsuka, Saitama, Japan**

Prove that

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+4}}\left(\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}-\frac{1}{F_{n+3}}\right) = \frac{1}{3}$;

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{F_nF_{n+4}}\left(\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}-\frac{1}{F_{n+3}}\right) = -\frac{1}{6}$;

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}}\left(\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}}-\frac{1}{F_{n+3}}\right) = \frac{1}{24}$.

**H-913 Proposed by Hideyuki Ohtsuka, Saitama, Japan**

Let $r \geq 1$ be an odd integer. Prove that there exist rational numbers $P_1$, $Q_1$, $P_2$, and $Q_2$ such that

$$\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(r-1)}{2}}}{F_nF_{n+1}F_{n+2}\cdots F_{n+r}} = P_1\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}+Q_1$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(F_nF_{n+1}F_{n+2}\cdots F_{n+r})^2} = P_2\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}+Q_2.$$

**H-914 Proposed by Benjamin Lee Warren, New York, NY**

Let $O_n = \frac{1}{3}n(2n^2+1)$ denote the $n$th Octahedral number and $C_n = \frac{1}{6}(n^3+5n+6)$ denote the $n$th Cake number. Prove the identity

$$C_{F_{2n}} + O_{F_{2n+1}} = C_{F_{2n+2}}.$$

**H-915 Proposed by the editor**

Prove that for all $k, m, n \geq 0$,

$$\sum_{j=0}^{n+2} \binom{n+2}{j} F_{2kj+m} = (L_{2k}+2) \sum_{j=0}^{n} \binom{n}{j} F_{2k(j+1)+m}$$

and

$$\sum_{j=0}^{n+2} \binom{n+2}{j} L_{2kj+m} = (L_{2k}+2) \sum_{j=0}^{n} \binom{n}{j} L_{2k(j+1)+m}.$$

## SOLUTIONS

**H-878 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 59, No. 3, August 2021)**

Prove that for all $n \geq 1$,

$$\sum_{k=1}^{n} L_k^3 L_{k+1}^3 = \frac{1}{9}\left(\left(\frac{5}{2}L_{3(n+1)} - L_{n+1}^3\right)^2 - 81\right).$$

**Solution by Hideyuki Ohtsuka, Saitama, Japan**

We have

$$\begin{aligned}
3L_rL_{r-1}L_{r+1} - 5L_{3r} &= 3L_r(\alpha^{r-1}+\beta^{r-1})(\alpha^{r+1}+\beta^{r+1}) - 5(\alpha^{3r}+\beta^{3r}) \\
&= 3L_r\left((\alpha^r+\beta^r)^2 - 2(\alpha\beta)^r + (\alpha\beta)^{r-1}(\alpha^2+\beta^2)\right) \\
&\quad -5\left((\alpha^r+\beta^r)^3 - 3(\alpha\beta)^r(\alpha^r+\beta^r)\right) \\
&= 3L_r(L_r^2 - 2(-1)^r - (-1)^r L_2) - 5(L_r^3 - 3(-1)^r L_r) \\
&= -2L_r^3.
\end{aligned}$$

Thus,

$$L_rL_{r-1}L_{r+1} = \frac{5L_{3r} - 2L_r^3}{3}$$

and using the solution of B-1247,

$$\begin{aligned}
\sum_{k=1}^{n} L_k^3 L_{k+1}^3 &= \left(\frac{L_nL_{n+1}L_{n+2}}{2}\right)^2 - \left(\frac{L_0L_1L_2}{2}\right)^2 \\
&= \frac{1}{4}(L_nL_{n+1}L_{n+2})^2 - 9 \\
&= \frac{1}{4}\frac{(5L_{3(n+1)} - 2L_{n+1}^3)^2}{9} - 9 \\
&= \frac{1}{9}\left(\left(\frac{5}{2}L_{3(n+1)} - L_{n+1}^3\right)^2 - 81\right).
\end{aligned}$$

**Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Wei-Kai Lai, Ángel Plaza, Raphael Schumacher, Albert Stadler, David Terr, Andrés Ventas, and the proposer.**

**<u>H-879</u> Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 59, No. 3, August 2021)**

Prove the following identities for the Fibonacci and Lucas numbers.

$$\begin{aligned}\sqrt{5}(F_{2n}-F_n) &= \sum_{k=1}^{n}\binom{n}{k}(k+1)^{k-1}((\alpha-k)^{n-k}-(\beta-k)^{n-k})\\ &= \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}(k-1)^{k-1}((\alpha+k)^{n-k}-(\beta+k)^{n-k}),\end{aligned}$$

and

$$\begin{aligned}L_{2n}-L_n &= \sum_{k=1}^{n}\binom{n}{k}(k+1)^{k-1}((\alpha-k)^{n-k}+(\beta-k)^{n-k})\\ &= \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}(k-1)^{k-1}((\alpha+k)^{n-k}+(\beta+k)^{n-k}).\end{aligned}$$

**Solution by Michel Bataille, Rouen, France**

We use the following result from [1]: If $n \in \mathbb{N}$ and $x, y \in \mathbb{C}$ with $x \neq 0$, then

$$\sum_{k=1}^{n}\binom{n}{k}(x+k)^{k-1}(y+n+1-k)^{n-k} = \frac{(x+y+n+1)^n-(y+n+1)^n}{x}. \qquad (1)$$

In (1), we let $x = 1$ and $y = \alpha - n - 1$ and obtain

$$\sum_{k=1}^{n}\binom{n}{k}(k+1)^{k-1}(\alpha-k)^{n-k} = \frac{(1+\alpha)^n-\alpha^n}{1} = \alpha^{2n}-\alpha^n.$$

Similarly,

$$\sum_{k=1}^{n}\binom{n}{k}(k+1)^{k-1}(\beta-k)^{n-k} = \beta^{2n}-\beta^n$$

and by subtraction and addition, the first and third identity follow. Next, we let $x = -1$ and $y = -\alpha - n - 1$ in (1) and deduce that

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}(k-1)^{k-1}(\alpha+k)^{n-k} = \alpha^{2n}-\alpha^n$$

and

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}(k-1)^{k-1}(\beta+k)^{n-k} = \beta^{2n}-\beta^n$$

and the second and fourth identity immediately follow.

## Reference

[1] M. Bataille, *Focus on … No. 15, A formula of Euler*, Crux Mathematicorum, **41.1** (2015), 16–18.

**Also solved by Dmitry Fleischman, Albert Stadler, and the proposer.**

**<u>H-880</u> Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain (Vol. 59, No. 3, August 2021)**

For any positive integer $k$, the Fibonacci $k$-sequence $\{F_{k,n}\}_{n\geq 0}$ is defined by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1$ with $F_{k,0} = 0$ and $F_{k,1} = 1$. Prove that

$$\sum_{i=0}^{n} \binom{2n+1}{n-i} F_{k,2i+1} = (k^2+4)^n.$$

**Solution by Albert Stadler, Herrliberg, Switzerland**

Binet's formula for the $k$-Fibonacci numbers is

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2},$$

where $\sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \frac{k-\sqrt{k^2+4}}{2}$. Clearly, $\sigma_1\sigma_2 = -1$. Thus,

$$\begin{aligned}
\sum_{i=0}^{n} \binom{2n+1}{n-i} F_{k,2i+1} &= \frac{1}{\sigma_1 - \sigma_2} \sum_{i=0}^{n} \binom{2n+1}{n-i} \left(\sigma_1^{2i+1} + \frac{1}{\sigma_1^{2i+1}}\right) \\
&= \frac{1}{\sqrt{k^2+4}} \sum_{i=0}^{n} \binom{2n+1}{n-i} \sigma_1^{2i+1} + \frac{1}{\sqrt{k^2+4}} \sum_{i=0}^{n} \binom{2n+1}{n+1+i} \frac{1}{\sigma_1^{2i+1}} \\
&= \frac{1}{\sqrt{k^2+4}} \sum_{i=0}^{n} \binom{2n+1}{i} \sigma_1^{2n+1-2i} + \frac{1}{\sqrt{k^2+4}} \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \frac{1}{\sigma_1^{2(i-n-1)+1}} \\
&= \frac{1}{\sqrt{k^2+4}} \sum_{i=0}^{2n+1} \binom{2n+1}{i} \sigma_1^{2n+1-2i} = \frac{1}{\sqrt{k^2+4}} \left(\sigma_1 + \frac{1}{\sigma_1}\right)^{2n+1} \\
&= \frac{1}{\sqrt{k^2+4}} \left(\sqrt{k^2+4}\right)^{2n+1} = (k^2+4)^n.
\end{aligned}$$

**Also solved by Michel Bataille, Brian Bradie, Nandan Sai Dasireddy, Dmitry Fleischman, Andrés Ventas, and the proposers.**

**<u>H-881</u> Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 59, No. 3, August 2021)**

For any positive integers $r$ and $n$, prove that

$$\sum_{k=0}^{n} \binom{2n}{n-k} \frac{F_{4rk}}{F_{4r}} = \sum_{k=0}^{n-1} \binom{2k}{k} L_{2r}^{2n-2k-2}.$$

**Solution by Albert Stadler, Herrliberg, Switzerland**

We express the binomial coefficient $\binom{2n}{n-k}$ as a complex integral using Cauchy's integral theorem. That is,

$$\binom{2n}{n-k} = \frac{1}{2\pi i} \int_{|z|=\frac{1}{2\alpha^{4r}}} \frac{(1+z)^{2n}}{z^{n-k+1}}\, dz,$$

where $i = \sqrt{-1}$ and $|z| = \frac{1}{2\alpha^{4r}}$ denotes the circle centered at the origin with radius $\frac{1}{2\alpha^{4r}}$ that is run through once in the positive direction. Then,

$$\begin{aligned}
\sum_{k=0}^{n}\binom{2n}{n-k}\frac{F_{4rk}}{F_{4r}} &= \frac{1}{\sqrt{5}F_{4r}}\sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{|z|=\frac{1}{2\alpha^{4r}}}\frac{(1+z)^{2n}}{z^{n-k+1}}\left(\alpha^{4rk}-\beta^{4rk}\right)dz \\
&= \frac{1}{\sqrt{5}F_{4r}}\frac{1}{2\pi i}\int_{|z|=\frac{1}{2\alpha^{4r}}}\frac{(1+z)^{2n}}{z^{n+1}}\left(\frac{1}{1-\alpha^{4r}z}-\frac{1}{1-\beta^{4r}z}\right)dz \\
&= \frac{1}{2\pi i}\int_{|z|=\frac{1}{2\alpha^{4r}}}\frac{(1+z)^{2n}}{z^{n}}\left(\frac{1}{1-L_{4r}z+z^2}\right)dz \\
&= \frac{1}{2\pi i}\int_{|z|=\frac{1}{2\alpha^{4r}}}\frac{(1+z)^{2n}}{z^{n}}\left(\frac{1}{(1+z)^2-L_{2r}^2 z}\right)dz \\
&= \frac{1}{2\pi i}\int_{|z|=\frac{1}{2\alpha^{4r}}}\frac{(1+z)^{2n-2}}{z^{n}}\left(\frac{1}{1-\frac{L_{2r}^2 z}{(1+z)^2}}\right)dz \\
&= \sum_{k=0}^{\infty}L_{2r}^{2k}\frac{1}{2\pi i}\int_{|z|=\frac{1}{2\alpha^{4r}}}\frac{(1+z)^{2n-2-2k}}{z^{n-k}}dz \\
&= \sum_{k=0}^{\infty}L_{2r}^{2k}\binom{2n-2-2k}{n-k-1} \\
&= \sum_{k=0}^{n-1}L_{2r}^{2(n-1-k)}\binom{2k}{k}.
\end{aligned}$$

In the proof, we used the identity

$$L_{4r}=\alpha^{4r}+\beta^{4r}=(\alpha^{2r}+\beta^{2r})^2-2=L_{2r}^2-2.$$

**Also solved by Dmitry Fleischman, and the proposer.**

**H-882 Proposed by Robert Frontczak, Stuttgart, Germany**
**(Vol. 59, No. 3, August 2021)**

Prove the following identities for Fibonacci and Lucas numbers.

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{k+1}=\frac{F_{2n+1}+L_{2n+1}}{n+1} \quad\text{and}\quad \sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{(k+1)(k+2)}=\frac{F_{2n+2}+L_{2n+2}-2}{(n+1)(n+2)}.$$

**Solution by Andrés Ventas, Santiago de Compostela, Spain**

We provide a generalization to gibonacci sequences $G_{n+2}=G_{n+1}+G_n$ with arbitrary $G_0$ and $G_1$. We will need the following lemma.

**Lemma.** *For all $n\geq 1$, we have*

$$\sum_{k=1}^{n}\binom{n}{k}G_{k-1}=G_{2n-1}-\Delta \quad\text{where}\quad \Delta=G_1-G_0 \tag{1}$$

*and*

$$\sum_{k=2}^{n}\binom{n}{k}G_{k-2}=G_{2n-2}-\Delta_n \quad\text{where}\quad \Delta_n=G_1+(n-2)(G_1-G_0). \tag{2}$$

*Proof.* Both identities are elementary and can be proved by induction on $n$. □

Let $G_{1,n}$ and $G_{2,n}$ be two gibonacci sequences. Then,

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}\frac{G_{1,k}+G_{2,k}}{k+1} &= \frac{n+1}{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{1,k}+G_{2,k}}{k+1}\\
&= \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}\left(G_{1,k}+G_{2,k}\right)\\
&= \frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k}\left(G_{1,k-1}+G_{2,k-1}\right)\\
&= \frac{1}{n+1}\left(G_{1,2n+1}+\Delta_1+G_{2,2n+1}+\Delta_2\right),
\end{aligned}$$

where (1) was used in the last step. For $G_{1,n}=F_n$ and $G_{2,n}=L_n$, we have $\Delta_1=F_1-F_0=1$ and $\Delta_2=L_1-L_0=-1$. Similarly using (2),

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}\frac{G_{1,k}+G_{2,k}}{(k+1)(k+2)} &= \frac{(n+1)(n+2)}{(n+1)(n+2)}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{1,k}+G_{2,k}}{(k+1)(k+2)}\\
&= \frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}\binom{n+2}{k+2}\left(G_{1,k}+G_{2,k}\right)\\
&= \frac{1}{(n+1)(n+2)}\sum_{k=2}^{n+2}\binom{n+2}{k}\left(G_{1,k-2}+G_{2,k-2}\right)\\
&= \frac{1}{(n+1)(n+2)}\left(G_{1,2n+2}-\Delta_{1,n}+G_{2,2n+2}-\Delta_{2,n}\right).
\end{aligned}$$

For $G_{1,n}=F_n$ and $G_{2,n}=L_n$, we have $\Delta_{1,n}=F_1-n(F_1-F_0)=1-n$ and $\Delta_{2,n}=L_1-n(L_1-L_0)=1+n$, so that $-\Delta_{1,n}-\Delta_{2,n}=-2$.

**Also solved by Michel Bataille, Brian Bradie, Nandan Sai Dasireddy, Dmitry Fleischman, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher, Jason L. Smith, Albert Stadler, and the proposer.**

**Errata:** In Advanced Problem **H-906**, the correct expression for $D_n$ is $D_n=(1-(-1)^n)/2$.