ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany or by e-mail at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-916 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let r and s be positive odd integers. Prove that

$$\prod_{n=1}^{\infty} \frac{F_{2n-1}+F_r}{F_{2n-1}+F_s} = \alpha^{\frac{r^2-s^2}{4}}.$$

H-917 Proposed by Benjamin Lee Warren, New York, NY

Let $O_n = \frac{1}{3}n(2n^2+1)$ denote the nth Octahedral number and $T_n = \frac{1}{6}n(n+1)(n+2)$ denote the nth Tetrahedral number. Prove the identity

$$O_{F_{2n}} + T_{F_{2n-1}-1} = T_{F_{2n+1}-1}.$$

H-918 Proposed by Andrés Ventas, Santiago de Compostela, Spain

Prove that

$$\sum_{n=0}^{\infty} \Big(\frac{1}{(L_{6n}/2)} \frac{1}{L_{6n+2}} + \frac{1}{L_{6n+2}} \frac{1}{L_{6n+3}+(L_{6n}/2)} + \frac{1}{L_{6n+4}} \frac{1}{L_{6n+3}+(L_{6n}/2)} + \frac{1}{L_{6n+4}} \frac{1}{(L_{6n+6}/2)} \Big) = \frac{1}{\sqrt{5}}.$$

H-919 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania

(a) If $a > 0$, compute $\lim_{n\to\infty} \sqrt[n]{(2n-1)!!F_n}\,(\sqrt[n]{a}-1)$;
(b) If $a > 0$ and $(b_n)_{n\geq 1}$ is a positive real sequence with $\lim_{n\to\infty} b_{n+1}/(nb_n) = b > 0$, compute $\lim_{n\to\infty} \sqrt[n]{b_nF_n}\,(\sqrt[n]{a}-1)$;
(c) Compute $\lim_{n\to\infty} n^2 \sqrt[n]{n!F_n}\,\sin(1/n^3)$;
(d) Compute $\lim_{n\to\infty} n \sqrt[n]{(2n-1)!!F_n}\,\sin(1/n^2)$.

H-920 Proposed by the editor

For $m \geq 0$, prove that

$$\sum_{k=1}^{\infty}(\zeta(4k)-1)F_{4k+m} = \frac{F_m}{2} + \frac{L_{m+2}}{5} + \frac{\pi}{4\sqrt{5}}L_{m+1}\tan\frac{\sqrt{5}\pi}{2} - \frac{\pi}{8\sqrt{5}}L_{m+1}A - \frac{\pi}{8}F_{m+1}B$$

and

$$\sum_{k=1}^{\infty}(\zeta(4k)-1)L_{4k+m} = \frac{L_m}{2} + F_{m+2} + \frac{\sqrt{5}\pi}{4}F_{m+1}\tan\frac{\sqrt{5}\pi}{2} - \frac{\pi}{8}L_{m+1}B - \frac{\sqrt{5}\pi}{8}F_{m+1}A,$$

where

$$A = \coth(\pi\alpha) + \coth(\pi/\alpha) \quad \text{and} \quad B = \coth(\pi\alpha) - \coth(\pi/\alpha),$$

$\alpha = (1+\sqrt{5})/2$, and $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$, $\Re(s) > 1$ is the Riemann zeta function.

SOLUTIONS

H-883 Proposed by Kenneth B. Davenport, Dallas, PA (Vol. 59, No. 4, November 2021)

Prove that for all $n \geq 1$:

(a) $3\sum_{k=1}^{n} F_{2k} + 4\sum_{k=1}^{n} F_{2k}^3 = F_{2n+1}^3 - 1;$

(b) $5\sum_{k=1}^{n} F_{2k} + 15\sum_{k=1}^{n} F_{2k}^3 + 11\sum_{k=1}^{n} F_{2k}^5 = F_{2n+1}^5 - 1;$

(c) $7\sum_{k=1}^{n} F_{2k} + 35\sum_{k=1}^{n} F_{2k}^3 + 56\sum_{k=1}^{n} F_{2k}^5 + 29\sum_{k=1}^{n} F_{2k}^7 = F_{2n+1}^7 - 1.$

Solution by Hideyuki Ohtsuka, Saitama, Japan

Let $a = F_{2k+1}$ and $b = F_{2k-1}$. Then, we have $a - b = F_{2k}$ and by Cassini's identity, $ab = F_{2k}^2 + 1$.

(a) Because $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$, we have

$$F_{2k+1}^3 - F_{2k-1}^3 = F_{2k}^3 + 3(F_{2k}^2+1)F_{2k} = 4F_{2k}^3 + 3F_{2k}.$$

Therefore, we have

$$\sum_{k=1}^{n}\left(4F_{2k}^3 + 3F_{2k}\right) = \sum_{k=1}^{n}\left(F_{2k+1}^3 - F_{2k-1}^3\right) = F_{2n+1}^3 - 1.$$

(b) Because $a^5 - b^5 = (a-b)^5 + 5ab(a-b)^3 + 5a^2b^2(a-b)$, we have

$$F_{2k+1}^5 - F_{2k-1}^5 = F_{2k}^5 + 5(F_{2k}^2+1)F_{2k}^3 + 5(F_{2k}^2+1)^2F_{2k} = 11F_{2k}^5 + 15F_{2k}^3 + 5F_{2k}.$$

Therefore, we have

$$\sum_{k=1}^{n}\left(11F_{2k}^5 + 15F_{2k}^3 + 5F_{2k}\right) = \sum_{k=1}^{n}\left(F_{2k+1}^5 - F_{2k-1}^5\right) = F_{2n+1}^5 - 1.$$

(c) Because $a^7 - b^7 = (a-b)^7 + 7ab(a-b)^5 + 14a^2b^2(a-b)^3 + 7a^3b^3(a-b)$, we have

$$F_{2k+1}^7 - F_{2k-1}^7 = 29F_{2k}^7 + 56F_{2k}^5 + 35F_{2k}^3 + 7F_{2k}.$$

Therefore, we have

$$\sum_{k=1}^{n}\left(29F_{2k}^7 + 56F_{2k}^5 + 35F_{2k}^3 + 7F_{2k}\right) = \sum_{k=1}^{n}\left(F_{2k+1}^7 - F_{2k-1}^7\right) = F_{2n+1}^7 - 1.$$

Also solved by Michel Bataille, Brian Bradie, Charles K. Cook, Dmitry Fleischman, Won Kyun Jeong, Wei-Kai Lai, Ángel Plaza, Raphael Schumacher, Albert Stadler, David Terr, Andrés Ventas, Ryan Zielinski, and the proposer.

H-884 Proposed by Robert Frontczak, Stuttgart, Germany
(Vol. 59, No. 4, November 2021)

Prove that

(i) $\displaystyle\sum_{n=2}^{\infty}\coth^{-1}(\alpha^n - \alpha^{-n}) = \frac{1}{2}\ln((\alpha+1)(\alpha+2))$, $\displaystyle\sum_{n=1}^{\infty}\coth^{-1}(\alpha^{2n} - \alpha^{-2n}) = \frac{1}{2}\ln(\alpha^3)$,
and $\displaystyle\sum_{n=1}^{\infty}\coth^{-1}(\alpha^{2n+1} - \alpha^{-2n-1}) = \frac{1}{2}\ln\left(\frac{\alpha+2}{\alpha}\right)$.

(ii) Deduce from (a) the following series evaluations:
$\displaystyle\sum_{n=1}^{\infty}\coth^{-1}\left(\frac{L_{4n+2}-1}{2L_{2n+1}}\right) = \ln\left(\frac{\alpha+2}{\alpha}\right)$, $\displaystyle\sum_{n=1}^{\infty}\coth^{-1}\left(\frac{L_{4n}-1}{2\sqrt{5}F_{2n}}\right) = 3\ln\alpha$,
and $\displaystyle\sum_{n=2}^{\infty}\coth^{-1}\left(\beta^n - \beta^{-n}\right) = \frac{1}{2}\ln((\alpha+1)(\alpha+2)) - 3\ln\alpha$.

Solution by Michel Bataille, Rouen, France

We have $\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ for $|x| > 1$ and for $n \in \mathbb{N}$,

$$\frac{\alpha^n - \alpha^{-n} + 1}{\alpha^n - \alpha^{-n} - 1} = \frac{\alpha^{2n} + \alpha^n - 1}{\alpha^{2n} - \alpha^n - 1} = \frac{(\alpha^{n-1}+1)(\alpha^{n+1}-1)}{(\alpha^{n-1}-1)(\alpha^{n+1}+1)} = \frac{U_{n-1}}{U_{n+1}},$$

where $U_n = \frac{\alpha^n+1}{\alpha^n-1}$. Also, note that $\alpha^n - \alpha^{-n} > 1$ for all integers $n \geq 2$ (because $\alpha^n - \alpha^{-n} = \sqrt{5}F_n$ or L_n according as n is even or odd).

(i) Let N be an integer with $N > 2$. Then, we have

$$\sum_{n=2}^{N}\coth^{-1}(\alpha^n - \alpha^{-n}) = \frac{1}{2}\sum_{n=2}^{N}\ln\left(\frac{\alpha^n - \alpha^{-n} + 1}{\alpha^n - \alpha^{-n} - 1}\right) = \frac{1}{2}\ln\left(\prod_{n=2}^{N}\frac{U_{n-1}}{U_{n+1}}\right) = \frac{1}{2}\ln\left(\frac{U_1U_2}{U_NU_{N+1}}\right).$$

Because $\lim\limits_{N\to\infty} U_N = 1$ and $\alpha^2 = \alpha + 1$ (so that $(\alpha-1)(\alpha^2-1) = \frac{1}{\alpha}\cdot\alpha = 1$), we obtain

$$\sum_{n=2}^{\infty}\coth^{-1}(\alpha^n - \alpha^{-n}) = \frac{1}{2}\ln\left(\frac{\alpha+1}{\alpha-1}\cdot\frac{\alpha^2+1}{\alpha^2-1}\right) = \frac{1}{2}\ln((\alpha+1)(\alpha+2)).$$

Similarly,

$$\sum_{n=1}^{N}\coth^{-1}(\alpha^{2n} - \alpha^{-2n}) = \frac{1}{2}\ln\left(\prod_{n=1}^{N}\frac{U_{2n-1}}{U_{2n+1}}\right) = \frac{1}{2}\ln\left(\frac{U_1}{U_{2N+1}}\right),$$

from which we deduce that

$$\sum_{n=1}^{\infty} \coth^{-1}(\alpha^{2n} - \alpha^{-2n}) = \frac{1}{2} \ln\left(\frac{\alpha+1}{\alpha-1}\right) = \frac{1}{2}\ln(\alpha^3)$$

(because $\alpha^3(\alpha - 1) = \alpha^2(\alpha^2 - \alpha) = \alpha^2 = \alpha + 1$). In the same way,

$$\sum_{n=1}^{\infty} \coth^{-1}(\alpha^{2n+1} - \alpha^{-2n-1}) = \frac{1}{2}\ln(U_2) = \frac{1}{2}\ln\left(\frac{\alpha^2+1}{\alpha^2-1}\right) = \frac{1}{2}\ln\left(\frac{\alpha+2}{\alpha}\right).$$

(ii) It is easily checked that if $|x| > 1$, then $\coth^{-1}\left(\frac{x^2+1}{2x}\right) = 2\coth^{-1}(x)$. Because

$$\frac{L_{4n+2}-1}{2L_{2n+1}} = \frac{(\alpha^{2n+1} - \alpha^{-2n-1})^2 + 1}{2(\alpha^{2n+1} - \alpha^{-2n-1})},$$

it follows that

$$\sum_{n=1}^{\infty} \coth^{-1}\left(\frac{L_{4n+2}-1}{2L_{2n+1}}\right) = 2\sum_{n=1}^{\infty}(\alpha^{2n+1} - \alpha^{-2n-1}) = \ln\left(\frac{\alpha+2}{\alpha}\right).$$

Similarly, from

$$\frac{L_{4n}-1}{2\sqrt{5}F_{2n}} = \frac{(\alpha^{2n} - \alpha^{-2n})^2 + 1}{2(\alpha^{2n} - \alpha^{-2n})},$$

we deduce that

$$\sum_{n=1}^{\infty} \coth^{-1}\left(\frac{L_{4n}-1}{2\sqrt{5}F_{2n}}\right) = 2\sum_{n=1}^{\infty}\coth^{-1}(\alpha^{2n} - \alpha^{-2n}) = \ln(\alpha^3) = 3\ln(\alpha).$$

Lastly, from $\coth^{-1}(-x) = -\coth^{-1}(x)$ and $\beta^{2n} - \beta^{-2n} = -(\alpha^{2n} - \alpha^{-2n})$, $\beta^{2n+1} - \beta^{-2n-1} = \alpha^{2n+1} - \alpha^{-2n-1}$ (because $\beta = -\alpha^{-1}$), we obtain

$$\sum_{n=1}^{\infty} \coth^{-1}(\beta^{2n} - \beta^{-2n}) = -\frac{3}{2}\ln(\alpha), \quad \sum_{n=1}^{\infty} \coth^{-1}(\beta^{2n+1} - \beta^{-2n-1}) = \frac{1}{2}\ln\left(\frac{\alpha+2}{\alpha}\right).$$

Thus,

$$\begin{aligned}\sum_{n=2}^{\infty} \coth^{-1}(\beta^{n} - \beta^{-n}) &= \frac{1}{2}\ln(\alpha+2) - \frac{1}{2}\ln(\alpha) - \frac{3}{2}\ln(\alpha)\\ &= \frac{1}{2}\ln((\alpha+1)(\alpha+2)) - \frac{1}{2}\ln(\alpha+1) - 2\ln(\alpha)\\ &= \frac{1}{2}\ln((\alpha+1)(\alpha+2)) - 3\ln\alpha.\end{aligned}$$

Also solved by Brian Bradie, Dmitry Fleischman, Won Kyun Jeong, Albert Stadler, Séan M. Stewart, David Terr, Andrés Ventas, and the proposer.

H-885 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 59, No. 4, November 2021)

Show that

$$\sum_{i=1}^{\infty} H_{2i-r}^{(2)} \frac{1}{\alpha^{2i}} = \left(\frac{\alpha+5-r}{10}\right)\frac{\pi^2}{6} - \left(\frac{\alpha+3-r}{4}\right)\ln^2(\alpha) \quad \text{hold for} \quad r=0,1,$$

where $H_n^{(2)} = \sum_{m=1}^{n} 1/m^2$. Deduce from these two identities the known (but nontrivial) result

$$\sum_{i=1}^{\infty} \frac{1}{i^2\alpha^{2i}} = \frac{\pi^2}{15} - \ln^2(\alpha).$$

Solution by Albert Stadler, Herrliberg, Switzerland

Clearly,

$$\frac{1}{m^2} = -\int_0^1 x^{m-1}\ln x\,dx.$$

Hence,

$$H_{2i-r}^{(2)} = \sum_{m=1}^{2i-r}\frac{1}{m^2} = -\int_0^1 \frac{1-x^{2i-r}}{1-x}\ln x\,dx$$

and

$$\sum_{i=1}^{\infty} H_{2i-r}^{(2)}\frac{1}{\alpha^{2i}} = -\int_0^1 \frac{1}{1-x}\left(\frac{1}{\alpha^2-1} - \frac{x^{2-r}}{\alpha^2-x^2}\right)\ln x\,dx.$$

For $r=0$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} H_{2i}^{(2)}\frac{1}{\alpha^{2i}} &= -\int_0^1 \frac{1}{1-x}\left(\frac{1}{\alpha^2-1} - \frac{x^2}{\alpha^2-x^2}\right)\ln x\,dx \\ &= \frac{\alpha}{2(\alpha-1)}\int_0^1 \frac{1}{x-\alpha}\ln x\,dx + \frac{\alpha}{2(\alpha+1)}\int_0^1 \frac{1}{x+\alpha}\ln x\,dx,\end{aligned}$$

because, by partial fraction decomposition,

$$-\frac{1}{1-x}\left(\frac{1}{\alpha^2-1} - \frac{x^2}{\alpha^2-x^2}\right) = \frac{\alpha}{2(\alpha-1)(x-\alpha)} + \frac{\alpha}{2(\alpha+1)(x+\alpha)}.$$

For $r=1$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} H_{2i-1}^{(2)}\frac{1}{\alpha^{2i}} &= -\int_0^1 \frac{1}{1-x}\left(\frac{1}{\alpha^2-1} - \frac{x}{\alpha^2-x^2}\right)\ln x\,dx \\ &= \frac{1}{2(\alpha-1)}\int_0^1 \frac{1}{x-\alpha}\ln x\,dx - \frac{1}{2(\alpha+1)}\int_0^1 \frac{1}{x+\alpha}\ln x\,dx,\end{aligned}$$

because, by partial fraction decomposition,

$$-\frac{1}{1-x}\left(\frac{1}{\alpha^2-1} - \frac{x}{\alpha^2-x^2}\right) = \frac{1}{2(\alpha-1)(x-\alpha)} - \frac{1}{2(\alpha+1)(x+\alpha)}.$$

The integrals above may be expressed in terms of the dilogarithm. If $|a|>1$, then

$$\int_0^1 \frac{\ln x}{x-a}\,dx = -\frac{1}{a}\int_0^1 \frac{\ln x}{1-\frac{x}{a}}\,dx = -\frac{1}{a}\sum_{k=0}^{\infty}\frac{1}{a^k}\int_0^1 x^k\ln x\,dx = \sum_{k=1}^{\infty}\frac{1}{a^k k^2} = Li_2\left(\frac{1}{a}\right).$$

It is well-known that

$$Li_2\left(\frac{1}{\alpha}\right) = \frac{\pi^2}{10} - \ln^2\alpha \quad \text{and} \quad Li_2\left(-\frac{1}{\alpha}\right) = -\frac{\pi^2}{15} + \frac{1}{2}\ln^2\alpha.$$

So,

$$\begin{aligned}\sum_{i=1}^{\infty} H_{2i}^{(2)}\frac{1}{\alpha^{2i}} &= \frac{\alpha}{2(\alpha-1)}Li_2\left(\frac{1}{\alpha}\right) + \frac{\alpha}{2(\alpha+1)}Li_2\left(-\frac{1}{\alpha}\right)\\ &= \left(\frac{\alpha+5}{10}\right)\frac{\pi^2}{6} - \left(\frac{\alpha+3}{4}\right)\ln^2\alpha\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{\infty} H_{2i-1}^{(2)}\frac{1}{\alpha^{2i}} &= \frac{1}{2(\alpha-1)}Li_2\left(\frac{1}{\alpha}\right) - \frac{1}{2(\alpha+1)}Li_2\left(-\frac{1}{\alpha}\right)\\ &= \left(\frac{\alpha+4}{10}\right)\frac{\pi^2}{6} - \left(\frac{\alpha+2}{4}\right)\ln^2\alpha.\end{aligned}$$

Finally,

$$\sum_{i=1}^{\infty}\frac{1}{i^2\alpha^{2i}} = 4\sum_{i=1}^{\infty}\frac{1}{(2i)^2\alpha^{2i}} = 4\sum_{i=1}^{\infty}\left(H_{2i}^{(2)} - H_{2i-1}^{(2)}\right)\frac{1}{\alpha^{2i}} = \frac{\pi^2}{15} - \ln^2\alpha.$$

Also solved by Brian Bradie, Dmitry Fleischman, Lucía L. Pacios and Andrés Ventas (jointly), Séan M. Stewart, and the proposer.

H-886 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania
(Vol. 59, No. 4, November 2021)

If $a, b, c \in (0, \pi/2)$ and $n \geq 1$, prove that

(i) $$\frac{\tan a}{F_n\sin 2b + F_{n+1}\sin 2c} + \frac{\tan b}{F_n\sin 2c + F_{n+1}\sin 2a} + \frac{\tan c}{F_n\sin 2a + F_{n+1}\sin 2b} > \frac{3}{2F_{n+2}};$$

(ii) $$\frac{\tan a}{F_n^2\sin 2b + F_{n+1}^2\sin 2c} + \frac{\tan b}{F_n^2\sin 2c + F_{n+1}^2\sin 2a} + \frac{\tan c}{F_n^2\sin 2a + F_{n+1}^2\sin 2b} > \frac{3}{2F_{2n+1}}.$$

Solution by Ángel Plaza, Gran Canaria, Spain

Because $F_n + F_{n+1} = F_{n+2}$ and $F_n^2 + F_{n+1}^2 = F_{2n+1}$, both inequalities follow from the following more general inequality

$$\frac{\tan a}{x\sin 2b + y\sin 2c} + \frac{\tan b}{x\sin 2c + y\sin 2a} + \frac{\tan c}{x\sin 2a + y\sin 2b} \geq \frac{3}{2(x+y)}.$$

Note that, for $\alpha \in (0, \pi/2)$, $\tan\alpha > \alpha$ and $\sin 2\alpha < 2\alpha$. Thus, it is enough to prove that

$$\frac{a}{xb+yc} + \frac{b}{xc+ya} + \frac{c}{xa+yb} \geq \frac{3}{x+y},$$

and

$$\frac{a^2}{a(xb+yc)} + \frac{b^2}{b(xc+ya)} + \frac{c^2}{c(xa+yb)} \geq \frac{3}{x+y}.$$

By Bergstrom's inequality,

$$\begin{aligned}\sum_{cycl}\frac{a^2}{a(xb+yc)} &\geq \frac{\left(\sum_{cycl}a\right)^2}{(x+y)\sum_{cycl}ab}\\ &= \frac{\sum_{cycl}a^2+2\sum_{cycl}ab}{(x+y)\sum_{cycl}ab}\\ &\geq \frac{3\sum_{cycl}ab}{(x+y)\sum_{cycl}ab}=\frac{3}{x+y}.\end{aligned}$$

Also solved by Michel Bataille, Dmitry Fleischman, Wei-Kai Lai, Albert Stadler, and the proposers.

H-887 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania
(Vol. 59, No. 4, November 2021)

If $m \geq 1$ is an integer, compute $\lim_{n\to\infty} n^{\cos^2 F_m}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{\sin^2 F_m} - \left(\sqrt[n]{n!}\right)^{\sin^2 F_m}\right)$.

Solution by Andrés Ventas, Santiago de Compostela, Spain

The solution to this problem is known. The problem appeared in a generalized form in the paper [1]. If L denotes the limit, then

$$L=\frac{\sin^2 F_m}{e^{\sin^2 F_m}}.$$

Reference

[1] D. M. Bătineţu-Giurgiu and N. Stanicu, *New methods for calculations of some limits*, The Teaching of Mathematics, **XVI.2** (2013), 82–88.

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Ángel Plaza, Albert Stadler, and the proposers.

H-888 Proposed by José Luis Díaz-Barrero, Barcelona, Spain
(Vol. 59, No. 4, November 2021)

For any integer $n \geq 1$, prove that

$$\sqrt{6F_n^4+3L_n^4}+\sqrt{5F_n^4+4L_n^4}+\sqrt{7F_n^4+2L_n^4}\geq F_{n+3}^2.$$

First Solution by Andrés Ventas, Santiago de Compostela, Spain

The equality holds for $n=1$. Let $n \geq 2$. We have

$$\sqrt{6F_n^4+3L_n^4}+\sqrt{5F_n^4+4L_n^4}+\sqrt{7F_n^4+2L_n^4}>\sqrt{3L_n^4}+\sqrt{4L_n^4}+\sqrt{2L_n^4}>L_n^2+2L_n^2+L_n^2=4L_n^2$$

and $4L_n^2 \geq F_{n+3}^2$ because, for all $n \geq 2$,

$$2L_n=2F_{n-1}+2F_{n+1}\geq F_{n-2}+F_{n-1}+2F_{n+1}=F_{n+3}>1.$$

Second Solution by Brian Bradie, Newport News, VA

By Jensen's inequality,

$$\begin{aligned} \sqrt{6F_n^4 + 3L_n^4} &\geq 2F_n^2 + L_n^2, \\ \sqrt{5F_n^4 + 4L_n^4} &\geq \frac{5}{3}F_n^2 + \frac{4}{3}L_n^2, \quad \text{and} \\ \sqrt{7F_n^4 + 2L_n^4} &\geq \frac{7}{3}F_n^2 + \frac{2}{3}L_n^2, \end{aligned}$$

where equality holds in each case if and only if $F_n = L_n$, that is, if and only if $n = 1$. Combining the above inequalities

$$\sqrt{6F_n^4 + 3L_n^4} + \sqrt{5F_n^4 + 4L_n^4} + \sqrt{7F_n^4 + 2L_n^4} \geq 6F_n^2 + 3L_n^2,$$

so it suffices to show that

$$6F_n^2 + 3L_n^2 \geq F_{n+3}^2.$$

Now,

$$6F_n^2 + 3L_n^2 - F_{n+3}^2 = 8(F_{n+2} - 2F_{n+1})^2 \geq 0.$$

Equality holds here if and only if $F_{n+2} = 2F_{n+1}$, that is, if and only if $n = 1$.

Also solved by Michel Bataille, Dmitry Fleischman, Wei-Kai Lai, Ángel Plaza, Albert Stadler, and the proposer.