ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-773 Proposed by H. Ohtsuka, Saitama, Japan.

Let B_n be the Bernoulli numbers defined by the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n.$$

For integers $n\ge 0$ and $m\ge 0$, prove that

$$\sum_{k=0}^{n}\binom{2n}{2k}F_{2mk}B_{2(n-k)}=\frac{n}{\sqrt{5}}\left[2\sum_{r=1}^{L_m}(\alpha^m-r)^{2n-1}+L_{m(2n-1)}\right].$$

H-774 Proposed by G. C. Greubel, Newport News, VA.

1. Let $m\ge 0,\ p\ge 0$ be integers. Evaluate the series
$$\sum_{n=0}^{\infty}\frac{F_{n+p}L_{n+m}}{(n+p)!(n+m)!}$$
in terms of the Bessel functions.
2. Evaluate the case $m=p$ in terms of a series of modified Bessel functions of the first kind. Take the limiting case $m\to 0$.
3. Show that when $p=0$ the series is given by
$$\sum_{n=0}^{\infty}\frac{F_nL_{n+m}}{n!(n+m)!}=\frac{1}{\sqrt{5}}\left(I_m(2\alpha)-I_m(2\beta)\right)-F_mJ_m(2).$$

H-775 Proposed by H. Ohtsuka, Saitama, Japan.

Let c be any real number $c\ne 2,\ -L_{2^n}$ for $n\ge 0$. Let

$$\gamma_c=\sqrt{5}\prod_{n=1}^{\infty}\left(1+\frac{c}{L_{2^n}}\right)^{-1}.$$

Prove that

$$\sum_{k=1}^{\infty} \frac{1}{(L_2+c)(L_4+c)\cdots(L_{2^k}+c)} = \frac{\gamma_c + c - 3}{c^2 - c - 2}.$$

<u>H-776</u> Proposed by H. Ohtsuka, Saitama, Japan.

Determine

$$\text{(i)} \quad \sum_{n=0}^{\infty}(-1)^n \tan^{-1}\frac{1}{L_{3^n}} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \tan^{-1}\frac{1}{F_{2n}} \tan^{-1}\frac{1}{L_{2n}}.$$

SOLUTIONS

<u>Sums of Fibonacci Numbers with Indices Given by Quadratic Forms</u>

<u>H-742</u> Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 51, No. 3, August 2013)

For positive integers $n,\ m$ and p with $p < m$ find a closed form expression for

$$\sum_{k_1,\ldots,k_m=1}^{n} F_{2k_1}\cdots F_{2k_m} F_{2(k_1^2+\cdots+k_p^2-k_{p+1}^2-\cdots-k_m^2)}.$$

Solution by the proposer.

We have

$$\left(\sum_{k=1}^{n} \alpha^{2k^2} F_{2k}\right)^p \left(\sum_{k=1}^{n} \beta^{2k^2} F_{2k}\right)^{m-p} = \sum_{k_1,\ldots,k_m=1}^{n} \alpha^{2(k_1^2+\cdots+k_p^2-k_{p+1}^2-\cdots-k_m^2)} F_{2k_1}\cdots F_{2k_m}, \quad (1)$$

and

$$\left(\sum_{k=1}^{n} \beta^{2k^2} F_{2k}\right)^p \left(\sum_{k=1}^{n} \alpha^{2k^2} F_{2k}\right)^{m-p} = \sum_{k_1,\ldots,k_m=1}^{n} \beta^{2(k_1^2+\cdots+k_p^2-k_{p+1}^2-\cdots-k_m^2)} F_{2k_1}\cdots F_{2k_m}. \quad (2)$$

We have

$$\begin{aligned}
\sum_{k=1}^{n} \alpha^{2k^2} F_{2k} &= \sum_{k=1}^{n} \alpha^{2k^2}\left(\frac{\alpha^{2k}-\alpha^{-2k}}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\sum_{k=1}^{n}(\alpha^{2k(k+1)} - \alpha^{2(k-1)k}) \\
&= \frac{1}{\sqrt{5}}(\alpha^{2n(n+1)} - 1) = \alpha^{n(n+1)} \cdot \frac{\alpha^{n(n+1)} - \alpha^{-n(n+1)}}{\sqrt{5}} \\
&= \alpha^{n(n+1)} F_{n(n+1)}. \qquad (3)
\end{aligned}$$

Similarly,

$$\sum_{k=1}^{n} \beta^{2k^2} F_{2k} = \beta^{n(n+1)} F_{n(n+1)}. \quad (4)$$

Using (1), (2), (3) and (4), we have

$$\sum_{k_1,\ldots,k_m=1}^{n} F_{2k_1}F_{2k_2}\cdots F_{2k_m}F_{2(k_1^2+\cdots+k_p^2-k_{p+1}^2-\cdots-k_m^2)}$$
$$=\frac{1}{\sqrt{5}}\left\{\left(\sum_{k=1}^{n}\alpha^{2k^2}F_{2k}\right)^p\left(\sum_{k=1}^{n}\beta^{2k^2}F_{2k}\right)^{m-p}-\left(\sum_{k=1}^{n}\beta^{2k^2}F_{2k}\right)^p\left(\sum_{k=1}^{n}\alpha^{2k^2}F_{2k}\right)^{m-p}\right\}$$
$$=\frac{1}{\sqrt{5}}\left(\alpha^{pn(n+1)}F_{n(n+1)}^p\beta^{(m-p)n(n+1)}F_{n(n+1)}^{m-p}-\beta^{pn(n+1)}F_{n(n+1)}^p\alpha^{(m-p)n(n+1)}F_{n(n+1)}^{m-p}\right)$$
$$=\frac{1}{\sqrt{5}}(\alpha^{(2p-m)n(n+1)}-\beta^{(2p-m)n(n+1)})F_{n(n+1)}^m=F_{(2p-m)n(n+1)}F_{n(n+1)}^m.$$

Also solved by Dmitry Fleischman.

<u>On the Fermat Quotient Modulo p</u>

<u>H-743</u> Proposed by Romeo Meštrović, Kotor, Montenegro.
(Vol. 51, No. 4, November 2013)

Let $p\ge 5$ be a prime and $q_p(2)=(2^{p-1}-1)/p$ be the Fermat quotient of p to base 2. Prove that

$$q_p(2)\equiv -\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{(-3)^k}{k} \pmod{p}.$$

Solution by the proposer.

Since

$$1\pm i\sqrt{3}=2\left(\cos\frac{\pi}{3}\pm i\sin\frac{\pi}{3}\right),$$

applying the de Moivre's formula, we have

$$(1+i\sqrt{3})^p+(1-i\sqrt{3})^p=2^p\left(\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^p+\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^p\right)$$
$$=2^p\cdot 2\cos\frac{p\pi}{3}=2^p, \qquad (1)$$

where we used the fact that $p\ge 5$ is odd and so $\cos(p\pi/3)=1/2$.

On the other hand, by the binomial theorem, we obtain

$$(1+i\sqrt{3})^p+(1-i\sqrt{3})^p=\sum_{k=0}^{p}\binom{p}{k}(i\sqrt{3})^k+\sum_{k=0}^{p}\binom{p}{k}(-1)^k(i\sqrt{3})^k$$
$$=2\sum_{\substack{0\le k\le p-1\\ 2|k}}\binom{p}{k}(i\sqrt{3})^k=2\sum_{k=0}^{(p-1)/2}\binom{p}{2k}(i\sqrt{3})^{2k}$$
$$=2\sum_{k=1}^{(p-1)/2}\binom{p}{2k}(-3)^k+2. \qquad (2)$$

The equalities (1) and (2) obviously yield the identity

$$\sum_{k=1}^{(p-1)/2} \binom{p}{2k} (-3)^k = 2^{p-1} - 1. \tag{3}$$

By the identity $\binom{p}{2k} = \frac{p}{2k}\binom{p-1}{2k-1}$ with $k = 1, \ldots, (p-1)/2$, (3) becomes

$$\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{(-3)^k}{k} \binom{p-1}{2k-1} = \frac{2^{p-1}-1}{p} := q_p(2). \tag{4}$$

Finally, since

$$\begin{aligned} \binom{p-1}{2k-1} &= \frac{(p-1)(p-2)\cdots(p-(2k-1))}{(2k-1)!} \\ &\equiv \frac{(-1)(-2)\cdots(-(2k-1))}{(2k-1)!} \pmod{p} \\ &\equiv (-1)^{2k-1} \pmod{p} \equiv -1 \pmod{p}, \end{aligned}$$

substituting this into (4), we obtain the desired congruence.

Inequalities Involving Sums of Reciprocals of Fibonacci and Lucas Numbers

H-744 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.
(Vol. 51, No. 4, November 2013)

Prove that

$$(1)\quad e^{n+3-L_{n+2}} \le \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{L_k}\right)^n; \qquad (2)\quad e^{n+2-L_nL_{n+1}} \le \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{L_k^2}\right)^n;$$

$$(3)\quad e^{n+1-F_{n+2}} \le \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{F_k}\right)^n; \qquad (4)\quad e^{n-F_nF_{n+1}} \le \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{F_k^2}\right)^n.$$

Solution by Robinson Higuita, Medellín, Colombia.

First of all, we prove that if $x \ge 1$, then $ex \le e^x$, or equivalently, $e^{1-x} \le \frac{1}{x}$. Let $g(x) = e^x - ex$. Since $g'(x) = e^x - e > 0$ for $x > 1$, we have that g is increasing in $(1, \infty]$. It is easy to see that $e^x - ex \ge 0$ for $x \ge 1$. This implies that $ex \le e^x$ all $1 \le x$. Therefore, $e^{1-x} \le \frac{1}{x}$. From this and the inequality of arithmetic and geometric means, we have that for every sequence $\{x_k\}_{1\le k\le n}$, with $1 \le x_k$, it holds that

$$e^{n-\sum_{k=1}^n x_k} = e^{1-x_1}e^{1-x_2}\ldots e^{1-x_n} \le \prod_{k=1}^{n} \frac{1}{x_k} \le \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}\right)^n,$$

namely,

$$e^{n-\sum_{k=1}^n x_k} \le \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}\right)^n. \tag{1}$$

On the other hand, it is known that (see for example pages 70, 77 and 78 in [1])

$$\sum_{k=1}^{n} L_k = L_{n+2} - 3,\ \sum_{k=1}^{n} F_k = F_{n+2} - 1,\ \sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 \text{ and } \sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

Therefore, if in (1) we take $x_k = L_k$, $x_k = F_k$, $x_k = L_k^2$ and $x_k = F_k^2$, we obtain

$$e^{n-(L_{n+2}-3)} = e^{n-\sum_{k=1}^{n} L_k} \le \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{L_k}\right)^n,$$

$$e^{n-(F_{n+2}-1)} = e^{n-\sum_{k=1}^{n} F_k} \le \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{F_k}\right)^n,$$

$$e^{n-(L_nL_{n+1}-2)} = e^{n-\sum_{k=1}^{n} L_k^2} \le \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{L_k^2}\right)^n,$$

$$e^{n-F_nF_{n+1}} = e^{n-\sum_{k=1}^{n} F_k^2} \le \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{F_k^2}\right)^n,$$

respectively.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, Inc., New York, 2001.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Harris Kwong, Hideyuki Ohtsuka, and the proposers.

On a Trigonometric Equation

H-745 Proposed by Kenneth B. Davenport, PA.
(Vol. 51, No. 4, November 2013)

Prove that $(a^2-1)\cos(n+3)\theta - 2\sqrt{a}\cos n\theta = (a-1)^2\cos(n+1)\theta$, where a is the real number satisfying $a^3 = a^2 + a + 1$ and θ is given by $\cos\theta = (1-a)\sqrt{a}/2$.

This problem was withdrawn in Vol. 52, No. 1, February 2014. Meanwhile, Dmitry Fleischman, G. C. Greubel, Zbigniew Jakubczyk, Anastasios Kotronis, and the proposer had provided solutions.

An Identity with Fibonomial Coefficients

H-746 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 51, No. 4, November 2013)

Define the generalized Fibonomial coefficient $\binom{n}{k}_{F;m}$ by

$$\binom{n}{k}_{F;m} = \frac{F_{mn}F_{m(n-1)}\cdots F_{m(n-k+1)}}{F_{mk}F_{m(k-1)}\cdots F_m} \qquad \text{for} \quad 0 < k \le n$$

with $\binom{n}{0}_{F;m} = 1$ and $\binom{n}{k}_{F;m} = 0$ (otherwise). Let $\varepsilon_i = (-1)^{(m+1)i}$. For positive integers n, m and s prove that

$$\sum_{i+j=2s} \varepsilon_i \binom{n}{i}_{F;m} \binom{n}{j}_{F;m} = \varepsilon_s \binom{n}{s}_{F;2m}.$$

Solution by the proposer.

Let $\binom{n}{k}_q$ be the q-binomial coefficient. The q-binomial theorem is given by

$$\sum_{k=0}^{n} x^k q^{k(k+1)/2} \binom{n}{k}_q = \prod_{k=1}^{n} (1 + xq^k). \tag{1}$$

Let $x = \alpha^{m(n+1)} z,\ q = (\beta/\alpha)^m = (-\alpha^{-2})^m$. We have

$$\begin{aligned}
\sum_{k=0}^{n} x^k q^{k(k+1)/2} \binom{n}{k}_q &= \sum_{k=0}^{n} \alpha^{mk(n+1)} z^k (-\alpha^{-2})^{mk(k+1)/2} \prod_{r=1}^{k} \frac{1-(\beta/\alpha)^{m(n-r+1)}}{1-(\beta/\alpha)^{mr}} \\
&= \sum_{k=0}^{n} (-1)^{mk(k+1)/2} \alpha^{mk(n+1)-mk(k+1)} z^k \prod_{r=1}^{k} \frac{\alpha^{m(n-r+1)} - \beta^{m(n-r+1)}}{\alpha^{mr} - \beta^{mr}} \cdot \alpha^{2mr-m(n+1)} \\
&= \sum_{k=0}^{n} (-1)^{mk(k+1)/2} z^k \prod_{r=1}^{k} \frac{F_{m(n-r+1)}}{F_{mr}} = \sum_{k=0}^{n} (-1)^{mk(k+1)/2} \binom{n}{k}_{F;m} z^k,
\end{aligned}$$

and

$$\prod_{k=1}^{n} (1 + xq^k) = \prod_{k=1}^{n} (1 + \alpha^{m(n+1)} (\beta/\alpha)^{mk} z) = \prod_{k=1}^{n} (1 + \alpha^{m(n-k+1)} \beta^{mk} z).$$

Therefore, by (1), we obtain

$$\sum_{k=0}^{n} (-1)^{mk(k+1)/2} \binom{n}{k}_{F;m} z^k = \prod_{k=1}^{n} (1 + \alpha^{m(n-k+1)} \beta^{mk} z). \tag{2}$$

Replacing z by $-z$ in (2) we get

$$\sum_{k=0}^{n} (-1)^{mk(k+1)/2+k} \binom{n}{k}_{F;m} z^k = \prod_{k=1}^{n} (1 - \alpha^{m(n-k+1)} \beta^{mk} z). \tag{3}$$

Using the identities (2) and (3), we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}_{F;2m}z^{2k} = \prod_{k=1}^{n}(1-\alpha^{2m(n-k+1)}\beta^{2mk}z^2)$$
$$= \prod_{k=1}^{n}(1-\alpha^{m(n-k+1)}\beta^{mk}z)(1+\alpha^{m(n-k+1)}\beta^{mk}z)$$
$$= \left(\sum_{i=0}^{n}(-1)^{mi(i+1)/2+i}\binom{n}{i}_{F;m}z^i\right)\left(\sum_{j=0}^{n}(-1)^{mj(j+1)/2}\binom{n}{j}_{F;m}z^j\right)$$
$$= \sum_{r=0}^{2n}\left(\sum_{i+j=r}(-1)^{(m/2)(i^2+i+j^2+j)+i}\binom{n}{i}_{F;m}\binom{n}{j}_{F;m}\right)z^r.$$

By comparing coefficients of z^{2s} we get

$$\sum_{i+j=2s}(-1)^{(m/2)(i^2+i+j^2+j)+i}\binom{n}{i}_{F;m}\binom{n}{j}_{F;m} = (-1)^s\binom{n}{s}_{F;2m}.$$

Here, since $i+j=2s$, we have

$$\frac{m}{2}(i^2+i+j^2+j) = \frac{m}{2}(i^2+i+(2s-i)^2+(2s-i)) = mi^2+2ms^2-2msi+ms.$$

Therefore, we have

$$\sum_{i+j=2s}(-1)^{mi+ms+i}\binom{n}{i}_{F;m}\binom{n}{j}_{F;m} = (-1)^s\binom{n}{s}_{F;2m},$$

which is the desired identity.

Solver's note: From the identity in this problem, we obtain the following identity easily:

$$\sum_{k=0}^{2n}\varepsilon_k\binom{2n}{k}_{F;m}^2 = \varepsilon_n\binom{2n}{n}_{F;2m}.$$

Moreover, we obtain the following identity in the same manner:

$$\sum_{f(a,a_1,\ldots,a_r)=2^r s}\varepsilon_{a,a_1}\binom{n}{a}_{F;m}\binom{n}{a_1}_{F;m}\binom{n}{a_2}_{F;2m}\cdots\binom{n}{a_r}_{F;2^{r-1}m} = (-1)^s\binom{n}{s}_{F;2^r m},$$

where $f(a,a_1,a_2,\ldots,a_r) = a+a_1+2a_2+\cdots+2^{r-1}a_r$ and $\varepsilon_{i,j} = (-1)^{m/2(i^2+i+j^2+j)+i}$.

Also partially solved by Dmitry Fleischman.