ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by April 15, 2009. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1044 Proposed by Paul S. Bruckman, Sointula, Canada

Prove the following identities:

(1) $(L_n)^2 = 2(F_{n+1})^2 - (F_n)^2 + 2(F_{n-1})^2$;

(2) $25(F_n)^2 = 2(L_{n+1})^2 - (L_n)^2 + 2(L_{n-1})^2$.

B-1045 Proposed by H.- J. Seiffert, Berlin, Germany

Show that, for all positive integers n,

$$\sum_{k=1}^{n} 4^{n-k} \frac{F_k}{F_{k+1}} \left(\prod_{j=k}^{n} \frac{F_j}{L_j} \right)^2 = \frac{F_n}{F_{n+1}}.$$

B-1046 Proposed by Michael Jemison (student), Northwest Missouri State University, Maryville, MO

Prove or disprove the following statement:
For $n \geq 1$,

$$F_{75(10^n)} \equiv 0 \pmod{10^{n+2}}.$$

SOLUTIONS

One Functional Identity

B-1034 Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA (Vol. 45.2, May 2007)

(a) Find a positive integer k and a polynomial $f(x)$ with rational coefficients such that

$$F_{kn} = f(L_n + f_n)$$

is an identity or prove that no identity of this form exists.
(b) Repeat part (a) with $f(L_n + F_n)$ replaced with $f(L_n - F_n)$.
(c) Repeat part (a) with $f(L_n + F_n)$ replaced with $f(L_n \times F_n)$.

Solution by H.-J. Seiffert, Berlin, Germany

(a) Such an identity cannot hold, since, otherwise $0 = F_0 = f(L_0 + F_0) = f(2) = f(L_1 + F_1) = F_k$, so that $k = 0$.
(b) Again, such an identity cannot exist, because, otherwise, $0 = F_0 = f(L_0 - F_0) = f(2) = f(L_2 - F_2) = F_{2k}$, giving $k = 0$.
(c) From equation (I_7) of [1], we know that $F_{2n} = L_n \times F_n$ for all n. Thus, the polynomial $f(x) = x$ has the desired property.

References

[1] V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, Santa Clara, CA, The Fibonacci Association, 1979.

Also solved by Paul S. Bruckman, Russell J. Hendel, and the proposer.

An Identity ... (mod 6)!

B-1035 Proposed by Hiroshi Matsui, Naoki Saita, Kazuki Kawata, Yusuke Sakurame, Toshiyuki Yamauchi, and Ryohei Miyadera, Kwansei Gakuin University, Nishinomiya, Japan (Vol. 45.2, May 2007)

Define $\{a_n\}$ by $a_1 = a_2 = 1$ and

$$a_n = a_{n-1} + a_{n-2} + \begin{cases} 1, & \text{if } n \equiv 1 \pmod{6}; \\ 0, & \text{if } n \not\equiv 1 \pmod{6} \end{cases}$$

for $n \geq 3$. Express a_n in terms of F_n.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Note that the recurrence relation holds for $n \geq 2$ if we set $a_0 = 0$. It is a routine exercise to show that

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-x^6} \cdot \frac{x}{1-x-x^2} = \left(\sum_{j=0}^{\infty} x^{6j}\right)\left(\sum_{i=0}^{\infty} F_i x^i\right).$$

Comparing coefficients of x^n yields

$$a_n = \sum_{\substack{i,j \geq 0 \\ i+6j=n}} F_i = \sum_{k=0}^{q} F_{6k+r}, \quad \text{where } \; n = 6q + r, 0 \leq r < 6.$$

Using Binet's formula, we find

$$\sqrt{5}\, a_n = \sum_{k=0}^{q} (\alpha^{6k+r} - \beta^{6k+r}) = \frac{\alpha^r(\alpha^{6q+6} - 1)}{\alpha^6 - 1} - \frac{\beta^r(\beta^{6q+6} - 1)}{\beta^6 - 1}.$$

Since $\alpha^6 - 1 = \alpha^6 + \alpha^3\beta^3 = \alpha^3(\alpha^3 + \beta^3) = 4\alpha^3$, and, in a similar manner, $\beta^6 - 1 = 4\beta^3$, we deduce that

$$4\sqrt{5}\, a_n = \alpha^{6q+r+3} - \beta^{6q+r+e} - (\alpha^{r-3} - \beta^{r-3}).$$

Therefore,

$$4a_n = F_{n+3} - F_{r-3} = \begin{cases} F_{n+3} - 2, & \text{if } n \equiv 0 \pmod 6; \\ F_{n+3} + 1, & \text{if } n \equiv 1 \pmod 6; \\ F_{n+3}, & \text{if } n \equiv 3 \pmod 6; \\ F_{n+3} - 1, & \text{otherwise.} \end{cases}$$

Also solved by Paul S. Bruckman, Charles K. Cook (Emeritus) and Michael R. Bacon (jointly), Russell J. Hendel, G. C. Greubel, H.-J. Seiffert, and the proposer.

An Inequality with Inverse Fibonacci Numbers

B-1036 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universitat Politécnica de Catalunya, Barcelona, Spain (Vol. 45.3, May 2007)

Let n be a positive integer. Prove that

$$\frac{1}{F_{n+2}} + \sum_{k=1}^{n} \left(\frac{1}{F_k^2} + \frac{1}{F_{k+1}^2}\right)^{1/2} < 1 + \sum_{k=1}^{n} \frac{1}{F_k}.$$

Solution by Paul S. Bruckman, Sointula, Canada

Let U_n and V_n represent the left and right side, respectively, of the given inequality, for $n \geq 1$. We seek to prove that $U_n < V_n$. Note that $U_1 = 1/2 + \sqrt{2} < 1.92$, while $V_1 = 2$; hence, $U_1 < V_1$. If we can prove that $U_{n+1} - U_n < V_{n+1} - V_n$ for $n \geq 1$, the result would follow by induction, since then $U_{n+1} < U_n - V_n + V_{n+1} < V_{n+1}$.

Now, $U_n - U_{n-1} = \frac{1}{F_{n+2}} - \frac{1}{F_{n+1}} + \frac{\sqrt{F_{2n+1}}}{F_n F_{n+1}}$, and $V_n - V_{n-1} = \frac{1}{F_n}$. Therefore,

$$D_n \equiv V_n - V_{n-1} - (U_n - U_{n-1}) = \frac{1}{F_n} + \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}} - \frac{\sqrt{F_{2n+1}}}{F_n F_{n+1}} = x + y - \frac{xy}{x+y} - \sqrt{x^2+y^2},$$

where $x = \frac{1}{F_n}$, $y = \frac{1}{F_{n+1}}$. Then $D_n = \frac{x^2+xy+y^2-(x+y)\sqrt{x^2+y^2}}{x+y}$. Now $(x^2+xy+y^2)^2 = x^4+2x^3y+3x^2y^2+2xy^3+y^4$, and $(x+y)^2(x^2+y^2) = x^4+2x^3y+2x^2y^2+2xy^3+y^4$, which shows that $D_n > 0$.

Also solved by G. C. Greubel, Russell J. Hendel, Carl Libis, and the proposer.

Fibonacci and Lucas Products Modulo A Prime

B-1037 Proposed by Vladimir Pletser, The Netherlands (Vol. 45.3, May 2007)

Let p be a prime. Prove or disprove each of the following.

(a) Except for $p = 5$, $p = 4r + 1$ divides $(F_{2r})(F_{2r+1})$.
(b) $p = 4r + 3$ divides $(L_{2r+1})(L_{2r+2})$.

Solution by Jaroslav Seibert, The Czech Republic

The relations $F_{2r}F_{2r+1} = 1/5(L_{4r+1} - 1)$, $L_{2r+1}L_{2r+2} = L_{4r+3} - 1$ are special cases of identities (17a), (17b) in [2]. Now it is sufficient to use the following well-known statement. Let p be a prime, then $L_p \equiv 1 \pmod{p}$ (see e.g. [1], Lemma 34.2, p. 410).

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.
[2] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Chichester: Ellis Horwood Ltd., 1989.

Remark. *The relations* $F_{2r}F_{2r+1} = 1/5(L_{4r+1} - 1)$, $L_{2r+1}L_{2r+2} = L_{4r+3} - 1$ *can be derived directly using the Binet formulas for the Fibonacci numbers and the Lucas numbers, respectively.*

Also solved by Paul S. Bruckman, G. C. Greubel, Herman Roelants, and the proposer.

A Matter of Regrouping!

B-1038 Proposed by the Problem Editor (Vol. 45.3, August 2007)

Prove or disprove:

$$F_{n+1}^4 + F_n^4 + 2F_n^3 + F_n^2 - F_{n+1}^2 - 2F_{n+1}^2F_n - 2F_{n+1}^2F_n^2$$

is divisible by $F_n + 2$ and F_{n-1} for all integers $n \geq 1$.

Solution by Charles K. Cook, Sumter, SC

Rewriting and factoring the given expression yields

$$\begin{aligned}F_{n+1}^4 &- 2F_{n+1}^2F_n^2 + F_n^4 + 2F_n(F_n^2 - F_{n+1}^2) + F_n^2 - F_{n+1}^2\\ &= (F_{n+1} + F_n)^2(F_{n+1} - F_n)^2 - 2F_n(F_{n+1} + F_n)(F_{n+1} - F_n) - (F_{n+1} + F_n)(F_{n+1} - F_n)\\ &= F_{n+2}F_{n-1}(F_{n+2}F_{n-1} - 2F_n - 1),\end{aligned}$$

proving the desired divisibility.

Also solved by Paul S. Bruckman, Russell J. Hendel, Rebecca A. Hillman, G. C. Greubel, Jaroslav Seibert, and the proposer.

A Sum of Inverse of Fibonacci Numbers

B-1039 Proposed by the Pantelimon George Popescu, Bucuresti, Romania and José Luis Díaz-Barrero, Barcelona, Spain (Vol. 45.3, August 2007)

Let n be a positive integer. Prove that

$$\frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \cdots + \frac{1}{F_nF_{n+2}} \geq \frac{2n^2}{F_nF_{n+1} + F_{n+2}^2 - 1}.$$

Solution by Paul S. Bruckman, Sointula, Canada

Suppose $\{a_n\}$ is any sequence of positive numbers. By the HM-AM inequality,

$$\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}} \leq \frac{1}{n}\sum_{k=1}^{n} a_k. \tag{1}$$

Letting $a_k = F_kF_{k+2}$, this yields,

$$\sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} \geq \frac{n^2}{\sum_{k=1}^{n} F_kF_{k+2}}. \tag{2}$$

We see that the given inequality will be proved if we can show that

$$\sum_{k=1}^{n} F_kF_{k+2} = \frac{1}{2}\{F_nF_{n+1} + F_{n+2}^2 - 1\}. \tag{3}$$

Let S_n and T_n represent the left and right members, respectively, of (3). First, note that $S_1 = F_1F_3 = 2 = \frac{1}{2}\{F_1F_2+F_3^2-1\} = \frac{1}{2}\{1+4-1\} = 2 = T_1$. Next, note that $S_n-S_{n-1} = F_nF_{n+2}$; on the other hand $T_n-T_{n-1} = \frac{1}{2}\{F_nF_{n+1}-F_{n-1}F_n+F_{n+2}^2-F_{n+1}^2\} = \frac{1}{2}\{F_n(F_{n+1}-F_{n-1})+(F_{n+2}-F_{n+1})(F_{n+2} + F_{n+1})\} = \frac{1}{2}\{F_n^2 + F_nF_{n+3}\} = \frac{F_n}{2}\{F_n + F_{n+1} + F_{n+2}\} = \frac{F_n}{2} \cdot 2F_{n+2} = F_nF_{n+2}$. Therefore, $S_n - S_{n-1} = T_n - T_{n-1}$, $n = 2, 3, \ldots$; since $S_1 = T_1$, it follows by induction that $S_n = T_n$ for all $n \geq 1$, which proves (3).

Also solved by Charles K. Cooke, G. C. Greubel, Russell J. Hendel, Rebecca A. Hillman, Jaroslav Seibert, and the proposer.

We would like to belatedly acknowledge the solutions to Problems B-1022 and B-1023 by Michael D. Hirschhorn, Problems B-1032 and B-1033 by James Sellers, and Problem B-1031 by Omprakash Sikhwal.