ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2014. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1131 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For integers a and b, prove that

$$\begin{aligned}&(F_a^2 + F_{a+1}^2 + F_{a+2}^2)(F_aF_b + F_{a+1}F_{b+1} + F_{a+2}F_{b+2})\\ &= 2(F_a^3F_b + F_{a+1}^3F_{b+1} + F_{a+2}^3F_{b+2});\end{aligned} \tag{1}$$

and

$$\begin{aligned}&(F_a^2 + F_{a+1}^2 + F_{a+2}^2)(F_aF_b^3 + F_{a+1}F_{b+1}^3 + F_{a+2}F_{b+2}^3)\\ &= (F_b^2 + F_{b+1}^2 + F_{b+2}^2)(F_a^3F_b + F_{a+1}^3F_{b+1} + F_{a+2}^3F_{b+2}).\end{aligned} \tag{2}$$

B-1132 **Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania**

Prove that

$$F_n^2 + F_{n+1}^2 + 5F_{n+2}^2 > 4\sqrt{6} \cdot \sqrt{F_n F_{n+1}} \cdot F_{n+2} \text{ for any positive integer } n; \tag{1}$$

and

$$L_n^2 + L_{n+1}^2 + 5L_{n+2}^2 > 4\sqrt{6} \cdot \sqrt{L_n L_{n+1}} \cdot L_{n+2} \text{ for any positive integer } n. \tag{2}$$

B-1133 **Proposed by Mohammad K. Azarian, University of Evansville, Indiana**

Determine the value of the following infinite series

$$S = \frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{3\cdot 8} - \frac{1}{5\cdot 8} + \frac{1}{8\cdot 13} + \frac{1}{8\cdot 21} - \frac{1}{13\cdot 21} + \frac{1}{21\cdot 34} + \frac{1}{21\cdot 55} - \frac{1}{34\cdot 55} + \frac{1}{55\cdot 89} + \frac{1}{55\cdot 144} - \frac{1}{89\cdot 144} + \frac{1}{144\cdot 233} + \frac{1}{144\cdot 377} - \cdots.$$

B-1134 **Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain and Francesc Gispert Sánchez, CFIS, Barcelona Tech, Barcelona, Spain**

Let n be a positive integer. Prove that

$$\frac{1}{F_n F_{n+1}}\left[\left(1-\frac{1}{n}\right)\sum_{k=1}^{n} F_k^{2n} + \prod_{k=1}^{n} F_k^2\right] \ge \left(\prod_{k=1}^{n} F_k^{(1-1/n)}\right)^2.$$

B-1135 **Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania**

Prove that

$$\frac{F_{n+1}^2}{F_n^3(F_nF_{n+1}+F_{n+2}^2)} + \frac{F_{n+2}^2}{F_{n+1}^3(F_{n+1}F_{n+2}+F_n^2)} + \frac{F_n^2}{F_{n+2}^3(F_nF_{n+2}+F_{n+1}^2)} > \frac{3}{2F_nF_{n+1}F_{n+2}}; \tag{1}$$

and

$$\frac{L_{n+1}^2}{L_n^3(L_nL_{n+1}+L_{n+2}^2)} + \frac{L_{n+2}^2}{L_{n+1}^3(L_{n+1}L_{n+2}+L_n^2)} + \frac{L_n^2}{L_{n+2}^3(L_nL_{n+2}+L_{n+1}^2)} > \frac{3}{2L_nL_{n+1}L_{n+2}}, \tag{2}$$

for any positive integer n.

SOLUTIONS

Fibonacci Numbers Divided by 11

B-1111 **Proposed by Mircea Merca, University of Craiova, Romania (Vol. 50.3, August 2012)**

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \left\lceil \frac{F_k}{11} \right\rceil = \left\lceil \frac{F_{n+2}-1}{11} + \frac{3n}{5} \right\rceil.$$

Solution by Marielle Silvio and Kasey Zemba (jointly) students at California University of Pennsylvania (CALURMA), California, PA.

It is enough to prove that

$$\sum_{k=1}^{n} \left\lceil \frac{F_k}{11} \right\rceil - 1 < \frac{F_{n+2}-1}{11} + \frac{3n}{5} \leq \sum_{k=1}^{n} \left\lceil \frac{F_k}{11} \right\rceil.$$

To show this, we prove that

$$\sum_{k=1}^{n} \left\lceil \frac{F_k}{11} \right\rceil = \frac{F_{n+2}-1}{11} + \frac{3n}{5} + l_n \tag{1}$$

for some rational number l_n such that $0 \leq l_n < 1$. We start with some preliminaries. We denote by $(n \bmod m)$ the remainder of n divided by m for $n, m \in \mathbb{N}$. Thus, the division algorithm can be written in the form

$$\left\lfloor \frac{n}{m} \right\rfloor = \frac{n}{m} - \frac{(n \bmod m)}{m}, \tag{2}$$

see [1, Id. 3.21, p. 82]. It is also known that

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n+m-1}{m} \right\rfloor \quad \text{and} \quad \sum_{k=1}^{n} F_k = F_{n+2} - 1, \tag{3}$$

see [1, Ex. 12, p. 96] and [2, Th. 5.1, p. 69], respectively. Using (2) and (3), we can easily see that

$$\begin{aligned}
\sum_{k=1}^{n} \left\lceil \frac{F_k}{11} \right\rceil &= \sum_{k=1}^{n} \left\lfloor \frac{F_k + 10}{11} \right\rfloor \\
&= \frac{1}{11} \sum_{k=1}^{n} (F_k + 10) - \frac{1}{11} \sum_{k=1}^{n} ((F_k + 10) \bmod 11) \\
&= \frac{F_{n+2}-1}{11} + \frac{10n}{11} - \frac{1}{11} \sum_{k=1}^{n} ((F_k + 10) \bmod 11). \qquad (4)
\end{aligned}$$

Since the Pisano period of $\{F_k\}_{k\in\mathbb{N}}$ modulo 11 is 10, see [3], the period of $\{F_k+10\}_{k\in\mathbb{N}}$ modulo 11 is also 10. Therefore,

$$\begin{aligned}\sum_{k=1}^{n}((F_k+10) \bmod 11) &= \left\lfloor\frac{n}{10}\right\rfloor\sum_{k=1}^{10}((F_k+10) \bmod 11)+\sum_{k=1}^{n \bmod 10}((F_k+10) \bmod 11)\\ &= 34\left\lfloor\frac{n}{10}\right\rfloor+\sum_{k=1}^{n \bmod 10}((F_k+10) \bmod 11)\\ &= 34\left(\frac{n}{10}-\frac{(n \bmod 10)}{10}\right)+\sum_{k=1}^{n \bmod 10}((F_k+10) \bmod 11), \qquad (5)\end{aligned}$$

where we used (2) in (5).

A straightforward computation from (4) and (5) shows that

$$\sum_{k=1}^{n}\left\lceil\frac{F_k}{11}\right\rceil=\frac{F_{n+2}-1}{11}+\frac{3n}{5}+l_n,$$

where

$$l_n=\frac{17}{55}(n \bmod m)-\frac{1}{11}\sum_{k=1}^{n \bmod 10}((F_k+10) \bmod 11).$$

Since $(n \bmod 10)\in\{0,\ldots 9\}$, one can easily see that $l_n\in\{0,\frac{17}{55},\frac{34}{55},\frac{46}{55},\frac{53}{55},\frac{10}{11},\frac{32}{55},\frac{4}{5},\frac{16}{55},\frac{3}{5}\}$. This proves (1). Thus, the proposed equality follows.

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, New York, 1990.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.
[3] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A001175.

Also solved by Paul S. Bruckman, Dmitry Fleishman, Russell Jay Hendel, David Stone and John Hawkins (jointly), and the proposer.

The Fifth and Seven Powers of Fibonacci Numbers

B-1112 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain (Vol. 50.3, August 2012)

Let n be a nonnegative integer. Show that

$$F_n^5+F_{n+1}^5+\frac{5}{7}\left(\frac{F_{n+2}^7-F_{n+1}^7-F_n^7}{F_{n+2}^2-F_nF_{n+1}}\right)$$

is a fifth perfect power.

Solution by Robinson Higuita (student), Universidad de Antioquia, Columbia.

From [1] we know that

$$F_{n+2}^5-F_n^5-F_{n+1}^5=5F_{n+2}F_{n+1}F_n[2F_{n+1}^2+(-1)^{n+1}] \qquad (1)$$
$$F_{n+2}^7-F_n^7-F_{n+1}^7=7F_{n+2}F_{n+1}F_n[2F_{n+1}^2+(-1)^{n+1}]^2. \qquad (2)$$

Dividing (1) by (2) we have

$$\frac{F_{n+2}^5 - F_n^5 - F_{n+1}^5}{F_{n+2}^7 - F_n^7 - F_{n+1}^7} = \frac{5}{7[2F_{n+1}^2 + (-1)^{n+1}]}.$$

So,

$$F_{n+2}^5 = F_n^5 + F_{n+1}^5 + \frac{5(F_{n+2}^7 - F_n^7 - F_{n+1}^7)}{7[F_{n+1}^2 + (F_{n+1}^2 + (-1)^{n+1})]}. \tag{3}$$

Using the Catalan Identity (see [2, p. 83]) we have

$$F_{n+2}F_n = F_{n+1}^2 + (-1)^{n+1}.$$

This and (3) imply that

$$F_{n+2}^5 = F_n^5 + F_{n+1}^5 + \frac{5(F_{n+2}^7 - F_n^7 - F_{n+1}^7)}{7[F_{n+1}^2 + F_{n+2}F_n]}. \tag{4}$$

Note that

$$F_{n+1}^2 + F_{n+2}F_n = F_{n+1}^2 + F_{n+2}(F_{n+2} - F_{n+1}) = F_{n+2}^2 + F_{n+1}(F_{n+1} - F_{n+2}) = F_{n+2}^2 - F_nF_{n+1}.$$

This and (4) imply that $F_n^5 + F_{n+1}^5 + \frac{5}{7}\left(\frac{F_{n+2}^7 - F_{n+1}^7 - F_n^7}{F_{n+2}^2 - F_nF_{n+1}}\right) = F_{n+2}^5$.

References

[1] L. Carlitz, *Problem H-112*, The Fibonacci Quarterly, **5.1** (1967), 71.

[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

Also solved by Brian D. Beasley, Paul S. Bruckman, Kenneth B. Davenport, Dmitry Fleischman, Russell Jay Hendel, Courtney Killian and Tricia Stoner (jointly), Harris Kwong, Kathleen Lewis, Carl Libis, Ángel Plaza, and the proposer.

Nesbitt Type Inequality with Fibonacci Numbers

B-1113 **Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania (Vol. 50.3, August 2012)**

Prove that

$$\frac{F_m^2}{(F_qF_n + F_{q+1}F_p)^2} + \frac{F_n^2}{(F_qF_p + F_{q+1}F_m)^2} + \frac{F_p^2}{(F_qF_m + F_{q+1}F_n)^2} \geq \frac{3}{F_{q+2}^2},$$

for any positive integers m, n, p, and q.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Since $F_{q+2} = F_{q+1} + F_q$, the proposed inequality is a particular case of the following more general inequality involving positive numbers:

$$\frac{x^2}{(ay + bz)^2} + \frac{y^2}{(az + bx)^2} + \frac{z^2}{(ax + by)^2} \geq \frac{3}{(a + b)^2}, \tag{1}$$

for any positive integers a, b, x, y, and z.

The last general inequality is a consequence of Chebyshev's sum inequality and of the following Nesbitt type inequality [1].

If a, b, x, y, and z are positive real numbers then

$$\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by}\geq\frac{3}{a+b}.$$

Then, by Chebyshev's sum inequality, the LHS of equation (1) satisfies

$$\begin{aligned}LHS &\geq \frac{\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by}}{3}\left(\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by}\right)\\ &\geq \frac{\left(\frac{3}{a+b}\right)^2}{3}\\ &= \frac{3}{(a+b)^2}.\end{aligned}$$

References

[1] S. Wu, O. Furdui, J. Seibert, P. Trojovský, *A note on a conjectured Nesbitt type inequality*, Taiwanese J. Math., **15.2** (2011), 449–456.

Also solved by Paul S. Bruckman, Dmitry Fleischman, and the proposer.

A Simple Inequality

B-1114 (Correction) Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania (Vol. 50.3, August 2012)

If $f : R^+ \to R^+$ such that $f(x) > x$ for all $x \in R^+$, prove that

$$\sum_{k=1}^{n} f((1+F_k^2)^2) > 4F_nF_{n+1}$$

for any positive integers n.

A composite solution by Amos G. Gera, Ashdod, Israel, and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain (independently).

We have

$$\begin{aligned}\sum_{k=1}^{n} f((1+F_k^2)^2) &> \sum_{k=1}^{n}(1+F_k^2)^2\\ &= n+2\sum_{k=1}^{n}F_k^2+\sum_{k=1}^{n}F_k^4\\ &\geq 4\sum_{k=1}^{n}F_k^2.\end{aligned}$$

Now, since $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, the conclusion follows.

Also solved by Paul S. Bruckman, Charles K. Cook, Dmitry Fleischman, and the proposer.

A Product Involving a Series With Inverse Fibonacci Numbers

B-1115 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 50.3, August 2012)

Given a positive integer m, prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} F_k^m\right)\left(\sum_{k=n+1}^{\infty}\frac{1}{F_k^m}\right) = \begin{cases} \frac{1}{L_m-2} & \text{(if } m \text{ is even)}, \\ \frac{1}{\sqrt{5}F_m-2} & \text{(if } m \text{ is odd)}. \end{cases}$$

Solution by the proposer.

We have

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{F_k^m}{F_n^m} &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{\alpha^k-\beta^k}{\alpha^n-\beta^n}\right)^m \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{\alpha^{k-n}-\beta^k\alpha^{-n}}{1-\beta^n\alpha^{-n}}\right)^m \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\alpha^{(k-n)m} \\
&= \lim_{n\to\infty}\sum_{j=0}^{n-1}\alpha^{-jm} = \sum_{j=0}^{\infty}\alpha^{-jm} \\
&= \frac{1}{1-\alpha^{-m}},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=n+1}^{\infty}\frac{F_n^m}{F_k^m} &= \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\left(\frac{\alpha^n-\beta^n}{\alpha^k-\beta^k}\right)^m \\
&= \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\left(\frac{1-\beta^n\alpha^{-n}}{\alpha^{k-n}-\beta^k\alpha^{-n}}\right)^m \\
&= \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\frac{1}{\alpha^{(k-n)m}} = \sum_{j=1}^{\infty}\frac{1}{\alpha^{jm}} \\
&= \frac{\alpha^{-m}}{1-\alpha^{-m}} = \frac{1}{\alpha^m-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left(\sum_{k=1}^{n}F_k^m\right)\left(\sum_{k=n+1}^{\infty}\frac{1}{F_k^m}\right) &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{F_k^m}{F_n^m}\right)\left(\sum_{k=n+1}^{\infty}\frac{F_n^m}{F_k^m}\right)\\ &= \frac{1}{1-\alpha^{-m}}\cdot\frac{1}{\alpha^m-1}\\ &= \frac{1}{\alpha^m+\alpha^{-m}-2} = \frac{1}{\alpha^m+(-\beta)^m-2}.\end{aligned}$$

The desired identity follows.

Also solved by Paul S. Bruckman and Dmitry Fleischman.

We would like to belatedly acknowledge Gera Amos for solving problem B-1108.