ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2012. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1096 **Proposed by Hideyuki Ohtsuka, Saitama, Japan**

For $n \geq 2$, prove that

$$\left(\sum_{k=1}^{n-1} \frac{L_k}{F_k}\right)\left(\sum_{k=1}^{n-1} \frac{1}{L_k L_{n-k}}\right) = \left(\sum_{k=1}^{n-1} \frac{F_k}{L_k}\right)\left(\sum_{k=1}^{n-1} \frac{1}{F_k F_{n-k}}\right).$$

B-1097 **Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain**

Compute the following sum

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{L_{n-1}(1+F_nF_{n+1}) - F_{n-1}(1+L_nL_{n+1})}{F_{2n-2} + (1+L_nL_{n+1})(1+F_nF_{n+1})}\right).$$

B-1098 **Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain**

Let n be a positive integer. Prove that

$$\left(\sum_{k=1}^{n} \frac{L_k^2}{\sqrt{1+L_k^2}}\right)\left(\prod_{k=1}^{n}(1+L_k^2)\right)^{1/2n} \leq L_nL_{n+1} - 2.$$

B-1099 **Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain**

For any positive integer k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$ and $L_{k,0} = 2$; $L_{k,1} = k$. Prove that

$$2^{n-1}F_{k,n} = \sum_{i\geq 0} k^{n-1-2i}(k^2+4)^i\binom{n}{2i+1}. \tag{1}$$

$$2^{n-1}L_{k,n} = \sum_{i\geq 0} k^{n-2i}(k^2+4)^i\binom{n}{2i}. \tag{2}$$

$$2^{n+1}F_{k,n+1} = \sum_{i=0}^{n} k^{n-i}2^iL_{k,i}. \tag{3}$$

B-1100 **Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain**

For any positive integer k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions

$F_{k,0} = 0$; $F_{k,1} = 1$, and $L_{k,0} = 2$; $L_{k,1} = k$. Prove that

$$\sum_{i\geq 0} \binom{2n}{i} F_{k,2i+1} = (k^2+4)^n F_{k,2n+1}. \tag{1}$$

$$\sum_{i\geq 0} \binom{2n+1}{i} F_{k,2i} = (k^2+4)^{n+1} L_{k,2n+1}. \tag{2}$$

$$\sum_{i\geq 0} \binom{2n}{i} L_{k,2i} = (k^2+4)^n L_{k,2n}. \tag{3}$$

$$\sum_{i\geq 0} \binom{2n+1}{i} L_{k,2i} = (k^2+4)^{n+1} F_{k,2n+1}. \tag{4}$$

SOLUTIONS

An "Inverse" Relation

B-1075 Proposed by Paul S. Bruckman, Nanaimo, BC, Canada (Vol. 48.3, August 2010)

The Fibonacci polynomials $F_n(x)$ may be defined by the following expression:

$$F_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k} \text{ for } n = 0, 1, 2, \ldots.$$

Prove the "inverse" relation:

$$x^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{n+1-2k}(x) \text{ for } n = 0, 1, 2, \ldots.$$

Solution by Ángel Plaza and Sergio Falcón (jointly) Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Las Palmas G. C. Spain

The Fibonacci plynomials may also be defined recurrently by $F_1(x) = 1$, $F_2(x) = x$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, if $n > 2$. (See [1, pp 443–446].)

Now, the proposed problem may be solved by induction. The equation is trivially true for $n = 0$. Let us suppose that it is true for every integer less than or equal to $n-1$, so

$$x^{n-1} = \sum_{k\geq 0} (-1)^k \binom{n-1}{k} F_{n-2k(x)}.$$

Now, by multiplying by x and keeping in mind that $xF_{n-2k}(x) = F_{n-2k+1}(x) - F_{n-2k-1}(x)$, we obtain

$$\begin{aligned} x^n &= \sum_{k=0}^{n} (-1)^k \binom{n-1}{k} xF_{n-2k}(x) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n-1}{k} (F_{n-2k+1}(x) - F_{n-2k-1}(x)) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n-1}{k} F_{n-2k+1}(x) - \sum_{k\geq 0} (-1)^k \binom{n-1}{k} F_{n-2k-1}(x) \\ &= \binom{n-1}{0} F_n(x) + \sum_{k=1}^{n} (-1)^k \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] F_{n-2k+1}(x) \\ &= \sum_{k=1}^{n} (-1)^k \binom{n}{k} F_{n-2k+1}(x). \end{aligned}$$

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, 2001.

Also solved by the proposer.

One of Many!

B-1077 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 48.4, November 2010)

Prove the following identity:

$$F_{n-2}^4 + L_n^4 + F_{n+2}^4 = 9(F_{n-1}^4 + F_n^4 + F_{n+1}^4).$$

Solution by George A. Hisert, Berkeley, California

Let the sequence $\{G_n\}$ be any sequence in which $G_{n+1} = G_n + G_{n-1}$ for all n. Then

a) $G_n = G_{n+1} - G_{n-1}$,
b) $G_{n+2} = G_{n+1} + G_n = G_{n+1} + (G_{n+1} - G_{n-1}) = 2G_{n+1} - G_{n-1}$, and
c) $G_{n-2} = G_n - G_{n-1} = (G_{n+1} - G_{n-1}) - G_{n-1} = G_{n+1} - 2G_{n-1}$.

And,

$$\begin{aligned} &9[(G_{n+1})^4 + (G_n)^4 + (G_{n-1})^4] - [(G_{n+2})^4 + (G_{n-2})^4] \\ &= 9[(G_{n+1})^4 + (G_{n+1} - G_{n-1})^4 + (G_{n-1})^4] - [(2G_{n+1} - G_{n-1})^4 + (G_{n+1} - 2G_{n-1})^4] \\ &= 9[(G_{n+1})^4 + (G_{n+1})^4 - 4(G_{n+1})^3(G_{n-1}) + 6(G_{n+1})^2(G_{n-1})^2 \\ &\quad - 4(G_{n+1})(G_{n-1})^3 + (G_{n-1})^4 + (G_{n-1})^4] - [16(G_{n+1})^4 - 32(G_{n+1})^3(G_{n-1}) \\ &\quad + 24(G_{n+1})^2(G_{n-1})^2 - 8(G_{n+1})(G_{n-1})^3 + (G_{n-1})^4 + (G_{n+1})^4 \\ &\quad - 8(G_{n+1})^3(G_{n-1}) + 24(G_{n+1})^2(G_{n-1})^2 - 32(G_{n+1})(G_{n-1})^3 + 16(G_{n-1})^4] \\ &= (G_{n+1})^4 + 4(G_{n+1})^3(G_{n-1}) + 6(G_{n+1})^2(G_{n-1})^2 + 4(G_{n+1})(G_{n-1})^3 + (G_{n-1})^4 \\ &= (G_{n+1} + G_{n-1})^4. \end{aligned}$$

If $G_n = F_n$, then using the identity $L_n = F_{n+1} + F_{n-1}$, we obtain the desired identity. If $G_n = L_n$, then using the identity $5F_n = L_{n+1} + L_{n-1}$, we obtain the identity

$$9(L_{n+1}^4 + L_n^4 + L_{n-1}^4) - (L_{n+2}^4 + L_{n-2}^4) = 625F_n^4;$$

which is the result of Problem B-1058.

Note: The solver obtained a much more general result, where 4 is replaced with a positive integer P and the coefficients 9 and 625 are adjusted accordingly.

Also solved by Brian Beasley, Paul S. Bruckman, Charles K. Cook, Kenneth B. Davenport, G. C. Greubel, Russell J. Hendel, Pedro Henrique O. Pantoja, Yashwant Kumar Panwar, Jaroslav Seibert, and the proposer.

Logarithmic Sum

B-1078 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Polytechnical University of Catalonia, Barcelona, Spain (Vol. 48.4, November 2010)

Let n be a nonnegative integer. Prove that

$$\frac{1}{n+1}\left(\sum_{k=0}^{n} \ln(1+F_k)\right)^2 \le F_nF_{n+1}.$$

Solution by Pedro Henrique O. Pantoja (student), University of Natal-RN, Brazil

For $x \ge 0$, it is well-known that $x \ge \ln(x+1)$. Thus,

$$\sum_{k=0}^{n} F_k \ge \sum_{k=0}^{n} \ln(1+F_k),$$

and so

$$\left(\sum_{k=0}^{n} F_k\right)^2 \ge \left(\sum_{k=0}^{n} \ln(1+F_k)\right)^2.$$

Since $\sum_{k=0}^{n} F_k^2 = F_nF_{n+1}$, by the Cauchy-Schwarz inequality,

$$(n+1)(F_nF_{n+1}) = \left(\sum_{k=0}^{n} 1\right)\left(\sum_{k=0}^{n} F_k^2\right) \ge \left(\sum_{k=0}^{n} F_k\right)^2 \ge \left(\sum_{k=0}^{n} \ln(1+F_k)\right)^2.$$

It follows that

$$\frac{1}{n+1}\left(\sum_{k=0}^{n} \ln(1+F_k)\right)^2 \le F_nF_{n+1}.$$

Also solved by Paul S. Bruckman, Sergio Falcón and Ángel Plaza (jointly), G. C. Greubel, Russell J. Hendel, Ohio Northern University Problem Solving Group, Jaroslav Seibert, and the proposer.

True But Not Strong Enough!

B-1079 **Proposed by Roman Witula, Silesian University of Technology, Poland (Vol. 48.4, November 2010)**

Prove or disprove the following statement:

$$5^n\left(F_{k-1}^{2n}+F_{k+1}^{2n}\right)\equiv\begin{cases}2(-1)^{k-1} & (\text{mod } L_k) \quad \text{if } n \text{ is odd,}\\ 2 & (\text{mod } L_k) \quad \text{if } n \text{ is even.}\end{cases}$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

We will establish the stronger statement:

$$5^n(F_{k+1}^{2n}+F_{k-1}^{2n})\equiv 2(-1)^{n(k-1)} \pmod{L_k^2}. \tag{4}$$

Thus, the statement of the problem is true but not strong enough, since the moduli L_k may be replaced by L_k^2. The starting point is the Waring Formula:

$$A^n+B^n=\sum_{j=0}^{[n/2]}\frac{n}{n-j}\binom{n-j}{j}(-AB)^j(A+B)^{n-2j}. \tag{5}$$

This is an identity valid for all non-negative integers n, and for all A and B. Replacing n by $2n$ gives,

$$A^{2n}+B^{2n}=\sum_{j=0}^{n}\frac{2n}{2n-j}\binom{2n-j}{j}(-AB)^j(A+B)^{2n-2j}. \tag{6}$$

Now set $A=F_{k+1}$, $B=F_{k-1}$. Note that $A+B=L_k$. Then,

$$\begin{aligned}F_{k+1}^{2n}+F_{k-1}^{2n}&=\sum_{j=0}^{n}\frac{2n}{2n-j}\binom{2n-j}{j}(-F_{k+1}F_{k-1})^jL_k^{2n-2j}\\ &\equiv\frac{2n}{2n-n}\binom{2n-n}{n}(-F_{k+1}F_{k-1})^n \pmod{L_k^2}.\end{aligned}$$

Then $5^n\{F_{k+1}^{2n}+F_{k-1}^{2n}\}\equiv 2(-5F_{k+1}F_{k-1})^n \pmod{L_k^2}$. Now $5F_{k+1}F_{k-1}=L_{2k}+3(-1)^k=L_k^2+(-1)^k$. Therefore, $5^n\{F_{k+1}^{2n}+F_{k-1}^{2n}\}\equiv 2(-1)^n\{L_k^2+(-1)^k\}^n \pmod{L_k^2}\equiv 2(-1)^n(-1)^{nk}$ $(\text{mod } L_k^2)$, which is equivalent to (1). We see that $2(-1)^{n(k-1)}=2(-1)^{k-1}$, if n is odd, or equal to 2, if n is even.

Also solved by G. C. Greubel and the proposer.

The Cubic Factor

B-1080 **Proposed by José Luis Díaz-Barrero, Barcelona, Spain (Vol. 48.4, November 2010)**

Let n be a nonnegative integer. Prove that

$$F_n^3(2F_n^2+5F_{n+1}F_{n+2})+F_{n+1}^3(2F_{n+1}^2+5F_nF_{n+2})=F_{n+2}^3(2F_{n+2}^2-5F_nF_{n+1}).$$

Solution by Jaroslav Seibert, University Paradubice, The Czech Republic

It is possible to prove the given equality for the so-called Gibonacci numbers G_n which satisfy the recurrence $G_{n+2} = G_{n+1} + G_n$ with arbitrary initial terms G_0, G_1. In fact,

$$\begin{aligned}
&G_n^3(2G_n^2 + 5G_{n+1}G_{n+2}) + G_{n+1}^3(2G_{n+1}^2 + 5G_nG_{n+2}) - G_{n+2}^3(2G_{n+2}^2 - 5G_nG_{n+1}) \\
&= G_n^3(2G_n^2 + 5(G_{n+2} - G_n)G_{n+2}) + (G_{n+2} - G_n)^3(2(G_{n+2} - G_n)^2 + 5G_nG_{n+2}) \\
&\quad - G_{n+2}^3(2G_{n+2}^2 - 5G_n(G_{n+2} - G_n)) \\
&= 2G_n^5 + 5G_n^3G_{n+2}^2 - 5G_n^4G_{n+2} + (G_{n+2}^3 - 3G_nG_{n+2}^2 + 3G_n^2G_{n+2} - G_n^3) \\
&\quad \cdot (2G_{n+2}^2 + G_nG_{n+2} + 2G_n^2) - 2G_{n+2}^5 + 5G_nG_{n+2}^4 - 5G_n^2G_{n+2}^3 \\
&= 2G_n^5 + 5G_n^3G_{n+2}^2 - 5G_n^4G_{n+2} + 2G_{n+2}^5 - 5G_nG_{n+2}^4 + 5G_n^2G_{n+2}^3 - 5G_n^3G_{n+2}^2 + 5G_n^4G_{n+2} \\
&\quad - 2G_n^5 - 2G_{n+2}^5 + 5G_nG_{n+2}^4 - 5G_n^2G_{n+2}^3 = 0.
\end{aligned}$$

The given identity immediately follows from this result.

Also solved by Paul S. Bruckman, Charles K. Cook, Kenneth B. Davenport, G. C. Greubel, Russell J. Hendel, George A. Hisert, Yashwant Kumar Panwar, Ángel Plaza, Moitland A. Rose, and the proposer.

We wish to acknowledge Kenneth B. Davenport for solving Problem B-1071.