ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2011. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2, \ \beta = (1-\sqrt{5})/2, \ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1076 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Find the closed form expression for

$$\prod_{k=1}^{n} (L_{2^{k+1}} - L_{2^k} + 1).$$

B-1077 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Prove the following identity:

$$F_{n-2}^4 + L_n^4 + F_{n+2}^4 = 9(F_{n-1}^4 + F_n^4 + F_{n+1}^4).$$

B-1078 **Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Polytechnical University of Catalonia, Barcelona, Spain.**

Let n be a nonnegative integer. Prove that

$$\frac{1}{n+1}\left(\sum_{k=0}^{n} ln(1+F_k)\right)^2 \leq F_n F_{n+1}.$$

B-1079 **Proposed by Roman Witula, Silesian University of Technology, Poland.**

Prove or disprove the following statement:

$$5^n\left(F_{k-1}^{2n} + F_{k+1}^{2n}\right) \equiv \begin{cases} 2(-1)^{k-1} & (\text{mod } L_k) \quad \text{if } n \text{ is odd,} \\ 2 & (\text{mod } L_k) \quad \text{if } n \text{ is even.} \end{cases}$$

B-1080 **Proposed by José Luis Díaz-Barrero, Barcelona, Spain.**

Let n be a nonnegative integer. Prove that

$$F_n^3(2F_n^2 + 5F_{n+1}F_{n+2}) + F_{n+1}^3(2F_{n+1}^2 + 5F_nF_{n+2}) = F_{n+2}^3(2F_{n+2}^2 - 5F_nF_{n+1}).$$

SOLUTIONS

Fibonacci, Lucas, and Pell Numbers Inequality

B-1056 **Proposed by Charles K. Cook, Sumter, SC (Vol. 46/47.4, November 2008/2009)**

If $n > 3$, show that

$$F_n^3 + L_n^3 + P_n^3 + 3F_nL_nP_n > 2(F_n + L_n)^2 P_n$$

where P_n is the nth Pell number.

Solution by Russell J. Hendell, Towson University, Towson, MD

Equivalently, upon expansion of the square on the right hand side of the problem identity, we must prove

$$F_n^3 + L_n^3 + P_n^3 > 2F_n^2P_n + 2L_n^2P_n + L_nF_nP_n.$$

Rearranging we equivalently must prove

$$P_n(P_n^2 - F_nL_n) > F_n^2(2P_n - F_n) + L_n^2(2P_n - L_n).$$

Since the problem identity is true for $n = 4$, and since

$$F_n^2(2P_n - F_n) + L_n^2(2P_n - L_n) < 2L_n^2(2P_n - L_n) < 4L_n^2P_n,$$

it suffices to prove for $n \geq 5$,

$$P_n(P_n^2 - F_nL_n) > 4L_n^2P_n.$$

Upon rearrangement, proof of this last assertion is equivalent to the proof of

$$\left(\frac{P_n}{L_n}\right)^2 - \frac{F_n}{L_n} > 4.$$

But by the Binet forms we have

$$F_n \sim \frac{1}{\sqrt{5}}\alpha^n, L_n \sim \alpha^n, P_n \sim \frac{\sqrt{2}}{4}(1+\sqrt{2})^n,$$

from which we infer that the left side of the last inequality is asymptotic to

$$O\left(\frac{1+\sqrt{2}}{\alpha}\right)^{2n} - O(1) \to \infty.$$

This completes the proof.

Also solved by Paul S. Bruckman and the proposer.

A Mod Power

B-1057 Proposed by Pat Costello, Eastern Kentucky University, Richmond, KY (Vol. 46/47.4, November 2008/2009)

For $n \geq 1$, prove that

$$2L_nL_{n-1} \equiv (-1)^{[\frac{n-1}{3}]}\left(5 - (-1)^{((n+2)\bmod 3)(\bmod 2)}\right) \pmod{10}$$

where $[x]$ is the greatest integer in x.

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

Given a sequence $\{u_n\}$ that is periodic, let $k(u_n)$ denote the period of this sequence. We assume that for all such sequences, $n = 1, 2, \ldots$.

Next, we note that $2L_nL_{n-1} = 2L_{2n-1} - 2(-1)^n$. The sequence $\{2L_{2n-1} \pmod{10}\}$ begins with $\{2, 8, \ldots\}$, with subsequent terms governed by the recurrence relation $u_{n+2} \equiv 3u_{n+1} - u_n \pmod{10}$. We then find that $\{2L_{2n-1} \pmod{10}\} = \{2, 8, 2, 8, 2, 8, \ldots\}$, or more briefly $\{2L_{2n-1} \pmod{10}\} = \{\overline{2, 8}\}$; note that $k\{2L_{2n-1} \pmod{10}\} = 2$. Also, $\{2(-1)^n\} = \{-2, 2, -2, 2, \ldots\}$, hence $\{2(-1)^n \pmod{10}\} = \{\overline{8, 2}\}$, also such that $k\{2(-1)^n \pmod{10}\} = 2$. Therefore, $k\{2L_nL_{n-1} \pmod{10}\} = 2$, and

$$\{2L_nL_{n-1} \pmod{10}\} = \{\overline{4, 6}\}. \tag{1}$$

Now observe that $\{[(n-1)/3]\} = \{0, 0, 0, 1, 1, 1, \ldots\}$ (a non-periodic sequence). Then $\{(-1)^{[(n-1)/3]}\} = \{1, 1, 1, -1, -1, -1, \ldots\}$. We see that $k\{(-1)^{[(n-1)/3]} \pmod{10}\} = 6$, and

$$\{(-1)^{[(n-1)/3]} \pmod{10}\} = \{\overline{1, 1, 1, 9, 9, 9}\}. \tag{2}$$

Likewise, we find that $\{n+2 \pmod 3\} = \{0, 1, 2, 0, 1, 2, \ldots\}$, and $\{n+2 \pmod 3 \pmod 2\} = \{\overline{0, 1, 0}\}$. Then $\{(-1)^{n+2 \pmod 3 \pmod 2}\} = \{\overline{1, -1, 1}\}$. Also, $\{5 - (-1)^{n+2 \pmod 3 \pmod 2}\} = \{\overline{4, 6, 4}\}$, and so:

$$\{5 - (-1)^{n+2 \pmod 3 \pmod 2} \pmod{10}\} = \{\overline{4, 6, 4}\}. \tag{3}$$

Multiplying (2) and (3), we obtain $k\{(-1)^{[(n-1)/3]}(5 - (-1)^{n+2 \pmod 3 \pmod 2}) \pmod{10}\} = 2$, with

$$\{(-1)^{[(n-1)/3]}(5 - (-1)^{n+2 \pmod 3 \pmod 2}) \pmod{10}\} = \{\overline{4, 6}\}. \tag{4}$$

Comparison of (1) and (4) yields the desired congruence relation.

Also solved by Russell J. Hendel and the proposer.

Two Identities for Quartic Fibonacci and Lucas Numbers

B-1058 Proposed by M. N. Despande, Nagpur, India (Vol. 46/47.4, November 2008/2009)

Prove the following identities:

(1) $9(F_{n+1}^4 + F_n^4 + F_{n-1}^4) - (F_{n+2}^4 + F_{n-2}^4) = L_n^4$;

(2) $9(L_{n+1}^4 + L_n^4 + L_{n-1}^4) - (L_{n+2}^4 + L_{n-2}^4) = 625F_n^4$.

Solution by Ángel Plaza and Sergio Falcón, jointly, Universidad de Las Palmas de Gran Canaria Las Palmas G. C., Spain

The identities may be proved by using Problem B-1044 (proposed by Paul S. Bruckman in The Fibonacci Quarterly, 46/47.1, February 2008/2009)

$$L_n^2 = 2F_{n+1}^2 - F_n^2 + 2F_{n-1}^2; \tag{3}$$

$$25F_n^2 = 2L_{n+1}^2 - L_n^2 + 2L_{n-1}^2. \tag{4}$$

Taking squares in (3) and using that $F_{n+1} = F_n + F_{n-1}$ it is obtained

$$L_n^4 = (2F_{n+1}^2 - F_n^2 + 2F_{n-1})^2 = F_n^4 + 16F_{n-1}^4 + 32F_nF_{n-1}^3 + 8F_n^3F_{n-1} + 24F_n^2F_{n-1}^2.$$

The same result is obtained from the left-hand side of (1), now also using that $F_{n-2} = F_n - F_{n-1}$.

The proof of (2) follows analogously from (4) using $L_{n+1} = L_n + L_{n-1}$ and $L_{n-2} = L_n - L_{n-1}$.

Also solved by Paul S. Bruckman, Charles K. Cook, G. C. Greubel, Russell J. Hendel, Geroge A. Hisert, Yashwant Kumar Panwar, Jarslav Seibert (two solutions), and the proposer.

Linear Combinations of Squares of Fibonacci and Lucas Numbers

B-1059 Proposed by George A. Hisert, Berkeley, CA (Vol. 46/47.4, November 2008/2009.)

For any positive integer r, find integers a, b, c and d such that

$$a(L_n)^2 = b(F_{n+r})^2 + c(F_n)^2 + d(F_{n-r})^2$$

and

$$25a(F_n)^2 = b(L_{n+r})^2 + c(L_n)^2 + d(L_{n-r})^2$$

for all positive integers n.

Solution by Jaroslav Seibert, Faculty of Economics and Administration, University of Pardubice, The Czech Republic

Consider the generalized Fibonacci numbers G_n defined by the recurrence $G_{n+2} = G_{n+1}+G_n$ with arbitrary initial terms. We will find integers a, b, c, and d such that the equality

$$a(G_{n-1} + G_{n+1})^2 = bG_{n+r}^2 + cG_n^2 + dG_{n-r}^2 \tag{1}$$

is valid for any positive integers r, n.

From identities (10a), (10b) in [1] it follows that

$$G_n = \frac{1}{L_r}(G_{n+r} + (-1)^r G_{n-r})$$

and

$$G_{n-1} + G_{n+1} = \frac{1}{F_r}(G_{n+r} - (-1)^r G_{n-r}).$$

After substituting in (1), we obtain

$$a\frac{1}{F_r^2}(G_{n+r} - (-1)^r G_{n-r})^2 = bG_{n+r}^2 + c\frac{1}{L_r^2}(G_{n+r} + (-1)^r G_{n-r})^2 + dG_{n-r}^2,$$

which can be rewritten in the form

$$\left(b + \frac{c}{L_r^2} - \frac{a}{F_r^2}\right) G_{n+r}^2 + 2(-1)^r \left(\frac{c}{L_r^2} + \frac{a}{F_r^2}\right) G_{n+r}G_{n-r} + \left(d + \frac{c}{L_r^2} - \frac{a}{F_r^2}\right) G_{n-r}^2 = 0.$$

This equality is valid if the three coefficients of the terms G_{n+r}^2, $G_{n+r}G_{n-r}$ and G_{n-r}^2 are equal to 0. This leads to the system

$$b + \frac{c}{L_r^2} - \frac{a}{F_r^2} = 0$$
$$\frac{c}{L_r^2} + \frac{a}{F_r^2} = 0$$
$$d + \frac{c}{L_r^2} - \frac{a}{F_r^2} = 0.$$

This system with unknowns a, b, c, d has infinitely many solutions. Choosing a as a parameter, the remaining unknowns can be expressed as $b = d = \frac{2a}{F_r^2}$, $C = -\frac{L_r^2}{F_r^2}a$, and hence the solution of the system is given by $(a, b, c, d) = (a, \frac{2a}{F_r^2}, -\frac{L_r^2}{F_r^2}, \frac{2a}{F_r^2})$.

As we want to find only integer solutions we can multiply every quadruple by F_r^2 and put a as an arbitrary integer. This way we obtain infinitely many quadruples $(a, b, c, d) = (F_r^2 a, 2a, -L_r^2 a, 2a)$ which satisfies Equation (1) independently of n.

Setting $G_n = F_n$ we have the first given identity and setting $G_n = L_n$ we get the second one (using identities (5) and (6) in [1]).

References

[1] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, Chichester, Ellis Horwood Ltd., 1989.

Also solved by Paul S. Bruckman, G. C. Greubel, Russell J. Hendel, and the proposer.

A Putative Inequality!

B-1060 Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, Universitat Politécnica de Cataluña, Barcelona, Spain (Vol. 46/47.4, November 2008/2009)

Let n be a positive integer. Prove that

$$1+\frac{1}{2}\left(\sum_{k=1}^{n} F_k 3^{1/F_k}+\sum_{k=1}^{n}\frac{L_k}{3^{1/L_k}}\right) > F_{2n+2}.$$

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

Let $S(n) = 1 + \frac{1}{2}\sum_{k=1}^{n}\{F_k 3^{1/F_k} + L_k 3^{-1/L_k}\}$, $n = 1, 2, \ldots$. The putative inequality states that $S(n) > F_{2n+2}$. However, it has been verified that this inequality is false for $n = 1, 2, \ldots, 20$. Based on the numerical evidence, it appears that the proposer intended the following problem. Prove that

$$S(n) > F_{n+3}, \quad \text{for} \quad n = 2, 3, \ldots. \tag{1}$$

We will prove the restated problem as indicated in (1), assuming this to be the correct formulation of the problem.

Note that $S(1) = 1 + \frac{1}{2}\left(3 + \frac{1}{3}\right) = \frac{8}{3} \approx 2.667$, while $F_4 = 3$, which shows that (1) does not hold for $n = 1$. Let $\mathcal{T}$ denote the set of natural numbers n such that the inequality indicated in (1) holds. Now $S(2) = S(1) + \frac{1}{2}\left(3 + 3^{2/3}\right) \approx 2.667 \approx +2.540 \approx 5.207 > 5 = F_5$. We then see that $2 \in \mathcal{T}$.

We note that $\lim 3^{1/F_n} = \lim 3^{-1/L_n} = 1$ as $n \to \infty$. Also note that $3^{1/F_n}$ approaches 1 from values greater than 1, while $3^{-1/L_n}$ approaches 1 from values less than 1.

Before we proceed, we consider the functions $x(3^{1/x} - 1)$ and $x(1 - 3^{-1/x})$. Make the substitution $x = 1/y$; then y is thought of as "small" as x grows large. Then $x(3^{1/x}-1) = \frac{3^y-1}{y}$. This is an increasing function of y, whose limit as $y \to 0^+$ is $\log 3 \approx 1.0986$. Thus, $x(3^{1/x} - 1)$ is a decreasing function of x; moreover, this function approaches $\log 3$ from above, as $x \to \infty$. Thus, $F_{n+1}(3^{\frac{1}{F_{n+1}}} - 1) > \log 3$.

Also, $x(1-3^{-1/x}) = \frac{1-3^{-y}}{y}$. This is a decreasing function of y, whose limit as $y \to 0^+$ is also $\log 3 \approx 1.0986$. This is therefore an increasing function of x that approaches its limit of $\log 3$ from below, as $x \to \infty$. Then we have $L_{n+1}(1 - 3^{-1/L_{n+1}}) < \log 3 < F_{n+1}(3^{1/F_{n+1}} - 1)$ for all $n \geq 1$. Clearly, the difference function $F_{n+1}(3^{1/F_{n+1}} - 1) - L_{n+1}(1 - 3^{-1/L_{n+1}})$ is a decreasing function of n; also, its limit as $n \to \infty$ is equal to zero, and it assumes only positive values. We are now prepared to proceed with an inductive proof of (1).

Suppose $n \in \mathcal{T}$ for some $n \geq 2$. That is, our inductive hypothesis is that (1) holds for some $n \geq 2$.

Then

$$\begin{aligned} S(n+1) &= S(n) + \frac{1}{2}\{F_{n+1}3^{1/F_{n+1}} + L_{n+1}3^{-1/L_{n+1}}\} \\ &= \frac{1}{2}\{F_{n+1} + L_{n+1}\} + \frac{1}{2}\{F_{n+1}(3^{1/F_{n+1}} - 1) - L_{n+1}(1 - 3^{-1/L_{n+1}})\}. \end{aligned}$$

Now, $\frac{1}{2}\{F_{n+1} + L_{n+1}\} = F_{n+2}$. Also as we previously showed, $\frac{1}{2}\{F_{n+1}(3^{1/F_{n+1}} - 1) - L_{n+1}(1 - 3^{-1/L_{n+1}})\} > 0$. That is $S(n+1) = S(n) + \frac{1}{2}\{F_{n+1} + L_{n+1}\} + \delta_n$, (where δ_n is small and positive). Then $S(n+1) = S(n) + F_{n+2} + \delta_n > F_{n+3} + F_{n+2} = F_{n+4}$. Thus, $n \in \mathcal{T} \Rightarrow (n+1) \in \mathcal{T}$. Since $2 \in \mathcal{T}$, the proof by induction is complete.

Also solved by the proposer.