ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2016. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

<u>B-1171</u> Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

For all integers $n \geq 1$, compute

$$\frac{(F_{n-1}+F_{n+1})^3 + (2F_n+F_{n+3})^3 + (5F_n+F_{n+3})^3 + (9F_n+F_{n+3})^3}{8F_{n+1}^3 + F_{n+3}^3 + (7F_n+F_{n+3})^3 + (8F_n+F_{n+3})^3}.$$

B-1172 Proposed by Steve Edwards, Kennesaw State University, Marietta, GA.

Show that the area of the triangle whose vertices have coordinates (F_n, F_{n+k}), (F_{n+2k}, F_{n+3k}), (F_{n+4k}, F_{n+5k}) is

$$\frac{5F_k^4 L_k}{2} \text{ if } k \text{ is even and } \frac{F_k^2 L_k^3}{2} \text{ if } k \text{ is odd.}$$

Also, find the area of the triangle whose vertices have coordinates (L_n, L_{n+k}), (L_{n+2k}, L_{n+3k}), (L_{n+4k}, L_{n+5k}).

B-1173 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

(i) Prove that

$$\frac{F_1}{(F_1^2+F_2^2)^{m+1}} + \frac{F_2}{(F_1^2+F_2^2+F_3^2)^{m+1}} + \cdots + \frac{F_n}{(F_1^2+F_2^2+\cdots+F_{n+1}^2)^{m+1}} \geq \frac{1}{F_{n+2}^m} - \frac{1}{F_{n+2}^{m+1}}$$

for any positive integer n and any positive real number m.

(ii) Prove that

$$\frac{L_1}{(L_1^2+L_2^2+2)^2} + \frac{L_2}{(L_1^2+L_2^2+L_3^2+2)^2} + \cdots + \frac{L_n}{(L_1^2+L_2^2+\cdots+L_{n+1}^2+2)^2} \geq \frac{(L_{n+2}-1)^2}{L_{n+2}^2(L_{n+2}-3)}$$

for any positive integer n.

B-1174 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{F_n^4 - 1} = -\frac{1}{18}.$$

B-1175 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Let $m \geq 0$ and $n \in N$. Prove that $\left(\sqrt{F_{2n+1}} - F_{n+1}\right)^m + \left(\sqrt{F_{2n+1}} + F_{n+1}\right)^m \geq 2F_n^m$.

SOLUTIONS

Radicals and Factorials!

B-1151 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania. (Vol. 52.3, August 2014)

Calculate each of the following:

(i) $\lim\limits_{n\to\infty} \left(\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n} \right)$,

(ii) $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!L_{n+1}} - \sqrt[n]{n!L_n}\right)$,

(iii) $\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!F_{n+1}} - \sqrt[n]{(2n-1)!!F_n}\right)$,

(iv) $\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!L_{n+1}} - \sqrt[n]{(2n-1)!!L_n}\right)$.

Solution by Hideyuki Ohtsuka, Saitama, Japan.

We use the following lemma.

Lemma. *(by Gh. Toader [1]). If the positive sequence $\{p_n\}$ is such that*

$$\lim_{n\to\infty}\frac{p_{n+1}}{np_n} = p > 0$$

then

$$\lim_{n\to\infty}(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}) = \frac{p}{e}.$$

(i) We have

$$\lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{n\cdot n!F_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{F_{n+1}}{F_n} = \alpha.$$

Therefore, using the lemma, we have

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n}\right) = \frac{\alpha}{e}.$$

(ii) Replacing F_n with L_n and using the lemma we get the same value for the limit in (i). Since

$$\lim_{n\to\infty}\frac{(2n+1)!!F_{n+1}}{n(2n-1)!!F_n} = \lim_{n\to\infty}\frac{(2+1)!!L_{n+1}}{n(2n-1)!!L_n} = \frac{2\alpha}{e},$$

the lemma implies that the limits in (iii) and (iv) have the same value $\frac{2\alpha}{e}$.

References

[1] Gh. Toader, *Lalescu sequences*, Publikacije-Elektrotehnickog Fakulteta Univerzitet U Beogradu Serija Matematika, 9 (1998), 19–28.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, G. C. Greubel, Harris Kwong, Ángel Plaza, and the proposer.

A Closed Form for an Infinite Sum

B-1152 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 52.3, August 2014)

For any positive integer number k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recurrently by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$ with respective initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. Find a closed form expression for

$$\sum_{n=1}^{\infty}\frac{F_{k,2^n}}{1+L_{k,2^{n+1}}}$$

as a function of $\alpha = \frac{k+\sqrt{k^2+4}}{2}$.

Solution by the proposers.

The sum for $k = 1$, that is for classical Fibonacci and Lucas numbers, is Example 7 in [1, p. 74]. The same argument given there may be applied for general k-Fibonacci and k-Lucas numbers. We use the following result from [1, Corollary 1].

Corollary. *Let $c, d \in \mathbb{Z}$, $d \geq 2$, $c \neq 0$. Let $P, Q \in \mathbb{C}[x]$ satisfying $P(0) = Q(0) = 1$ and $P(x^d) = P(x)Q(x)$. Then for every $|x| < 1$*

$$\sum_{n=0}^{\infty} \left(\frac{d}{c}\right)^n x^{d^n} \frac{Q'(x^{d^n})P(x^{d^n}) - (c-1)P'(x^{d^n})Q(x^{d^n})}{P(x^{d^{n+1}})} = -cx\frac{P'(x)}{P(x)}. \tag{1}$$

Taking $d = c = 2$, $P(x) = x^2 + x + 1$, and $Q(x) = x^2 - x + 1$ it is obtained

$$\sum_{n=0}^{\infty} x^{2n} \frac{x^{2^{n+1}} - 1}{x^{2^{n+2}} + x^{2^{n+1}} + 1} = -\frac{x(2x+1)}{x^2 + x + 1}. \tag{2}$$

Note that $F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_{k,n} = \alpha^n + \beta^n$ where $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta = \frac{k-\sqrt{k^2+4}}{2}$, so by letting $x = \alpha^{-1}$ in (2), we have

$$\begin{aligned} \frac{\alpha - \alpha^{-1}}{1 + \alpha^2 + \alpha^{-2}} + \sum_{n=1}^{\infty} \frac{(\alpha - \beta)F_{k,2^n}}{1 + L_{k,2^{n+1}}} &= \frac{1}{\alpha} \cdot \frac{\alpha + 2}{1 + \alpha + \alpha^{-1}} \\ \sum_{n=1}^{\infty} \frac{(\alpha - \beta)F_{k,2^n}}{1 + L_{k,2^{n+1}}} &= \frac{2 + \alpha^2}{1 + \alpha^2 + \alpha^4}. \end{aligned}$$

Since $\alpha - \beta = \alpha + \alpha^{-1}$, we obtain

$$\sum_{n=1}^{\infty} \frac{F_{k,2^n}}{1 + L_{k,2^{n+1}}} = \left(\alpha + \frac{1}{\alpha}\right)^{-1} \left(\frac{2 + \alpha^2}{1 + \alpha^2 + \alpha^4}\right).$$

References

[1] D. Duverneya and I. Shiokawa, *On series involving Fibonacci and Lucas numbers I*, AIP Conf. Proc. 976, March 5-7, 2007, Kyoto (Japan), Editor Takao Komatsu, (2008), 62–76.

Evaluate a Lucas Sum

B-1153 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 52.3, August 2014)

For any positive integer number k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recurrently by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$ with respective initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. Prove that

$$\sum_{i=0}^{n} \left(\frac{2}{k}\right)^i L_{k,i} = k\left(\frac{2}{k}\right)^{n+1} F_{k,n+1}.$$

Solution by Kenneth B. Davenport, Dallas, PA.

First, we point out that this result appears to be a generalization of identity 23 in [1],

$$\sum_{i=0}^{n} 2^i L_i = 2^{n+1} F_{n+1} \tag{1}$$

Thus, for $k = 1$ in the stated identity, (1) can be easily derived. In view of this, it may be better to solve

$$\sum_{i=0}^{n} x^i L_{k,i} \tag{2}$$

for a general x value and then let $x = \frac{2}{k}$.

In the Binet form, it is known that k-Fibonacci and k-Lucas sequences satisfy

$$F_{k,n} = \frac{a^n - \beta^n}{a - \beta}; L_{k,n} = a^n + \beta^n$$

where

$$a = \frac{k + \sqrt{k^2+4}}{2}; \beta = \frac{k - \sqrt{k^2+4}}{2}.$$

So, (2) is written

$$\sum_{i=0}^{n} x^i (a^i + \beta^i) = \sum_{i=0}^{n} [(ax)^i + (\beta x)^i]. \tag{3}$$

And we know that the sum of a geometric ratio, say r is

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}. \tag{4}$$

Using (4), we see that (3) is

$$\frac{1 - (ax)^{n+1}}{1 - ax} + \frac{1 - (\beta x)^{n+1}}{1 - \beta x}.$$

Combining the fractions and noting $a\beta = -1$; $a + \beta = k$; we get

$$\frac{2 - kx - (ax)^n (ax + x^2) - (\beta x)^n (\beta x + x^2)}{(1 - kx - x^2)}.$$

Now, letting $x = \frac{2}{k}$ we have, after some simplification,

$$-\left(\frac{2}{k}\right)^n \left[a^n \cdot \left(\frac{k + \sqrt{k^2+4}}{2} \cdot \frac{2}{k} + \frac{4}{k^2} \right) - \beta^n \cdot \left(\frac{k - \sqrt{k^2+4}}{2} \cdot \frac{2}{k} + \frac{4}{k^2} \right) \right]$$

and in the denominator we would have

$$-\left(\frac{k^2+4}{k^2} \right)$$

canceling the minus signs and factoring out $\sqrt{k^2+4}$ from the numerator then yields

$$\left(\frac{2}{k}\right)^n \cdot \left(\frac{a^n (k + \sqrt{k^2+4}) + \beta^n (-k + \sqrt{k^2+4})}{\sqrt{k^2+4}} \right).$$

So finally we can see this is now

$$k \left(\frac{2}{k}\right)^{n+1} F_{k,n+1}$$

thereby verifying the stated identity.

References

[1] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count. The Art of Combinatorial Proof*, Mathematical Assn. of America, Washington, DC, 2003.

Also solved by G. C. Greubel, Russell Jay Hendel, Harris Kwong, N. Padmaja (student), and the proposer.

Sum ... of Products ... of Squares ...

B-1154 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA. (Vol. 52.3, August 2014)

Find a closed form expression for

$$\sum_{i=0}^{n} L_i^2 L_{i+1}^2.$$

Solution by Kaige M. Lindberg, Charleston, SC.

From [1, p. 90 Identity 63] we know that $L_nL_{n+1} = L_{2n+1} + (-1)^n$. From here we can see that $(L_nL_{n+1})^2 = (L_{2n+1} + (-1)^n)^2$. Therefore,$\sum_{i=0}^{n}(L_iL_{i+1})^2 = \sum_{i=0}^{n}(L_{2i+1} + (-1)^i)^2$. Now it is easy to see that

$$\sum_{i=0}^{n} L_i^2 L_{i+1}^2 = \sum_{i=0}^{n} L_{2i+1}^2 + 2\sum_{i=0}^{n} L_{2i+1}(-1)^i + \sum_{i=0}^{n} 1. \tag{1}$$

We find the closed forms for the three summations which make up the right side of (1). The rightmost sum is, obviously, $\sum_{i=0}^{n} 1 = n + 1$. From [2, p. 32 Identity 54] it is easy to see that $\sum_{i=0}^{n} L_{2i+1}^2 = F_{4n+4} - 2n - 2$. And from [2, p. 32 Identity 55] it is also easy to see that $\sum_{i=0}^{n} L_{2i+1}(-1)i = F_{2n+2}(-1)^n$. By substituting the closed forms of these three different summations into (1) we get

$$\sum_{i=0}^{n} L_i^2 L_{i+1}^2 = F_{4n+4} + 2F_{2n+2}(-1)^n - n - 1.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers wth Applications*, John Wiley, New York, 2001.
[2] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count. The Art of Combinatorial Proof*, Mathematical Assn. of America, Washington, DC, 2003.

Also solved by Adnan Ali (student), Kenneth B. Davenport, Dmitry Fleischman, G. C. Greubel, Russell J. Hendel, Harris Kwong, Kathleen Lewis, Hideyuki Ohtsuka, Ángel Plaza, Ashley Reavis, Jason L. Smith, and the proposer.

Evaluate Two More Infinite Series

B-1155 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 52.3, August 2014)

Prove each of the following:

(i) $\displaystyle\sum_{k=1}^{\infty}\frac{L_{2^{k+1}}}{F_{3\cdot 2^k}}=\frac{5}{4}$,

(ii) $\displaystyle\sum_{k=1}^{\infty}\frac{F_{2^{k-1}}^2}{L_{2^k}^2-1}=\frac{3}{20}$.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

We first derive three preliminary results. It follows from $L_mL_n = L_{m+n} + (-1)^n L_{m-n}$ and $F_mF_n = L_{m+n} - (-1)^n L_{m-n}$ that for $k \geq 2$,

$$\begin{aligned}
L_{3\cdot 2^{k-1}}L_{2^{k-1}} &= L_{2^{k+1}} + L_{2^k},\\
5F_{2^{k-1}}^2 &= L_{2^k} - 2,\\
L_{2^{k-1}}^2 &= L_{2^k} + 2.
\end{aligned}$$

(i) Since $F_mL_m = F_{2m}$, we find for $k \geq 2$,

$$\frac{L_{2^{k+1}}}{F_{3\cdot 2^k}} = \frac{L_{3\cdot 2^{k-1}}L_{2^{k-1}} - L_{2^k}}{F_{3\cdot 2^k}} = \frac{L_{2^{k-1}}}{F_{3\cdot 2^{k-1}}} - \frac{L_{2^k}}{F_{3\cdot 2^k}}.$$

Its telescoping nature implies that

$$\sum_{k=1}^{\infty}\frac{L_{2^{k+1}}}{F_{3\cdot 2^k}} = \frac{L_4}{F_6} + \sum_{k=2}^{\infty}\left(\frac{L_{2^{k-1}}}{F_{3\cdot 2^{k-1}}} - \frac{L_{2^k}}{F_{3\cdot 2^k}}\right) = \frac{L_4}{F_6} + \frac{L_2}{F_6} = \frac{5}{4}.$$

(ii) We find for $k \geq 2$,

$$\begin{aligned}
5F_{2^{k-1}}^2(L_{2^{k-1}}^2 - 1) &= (L_{2^k} - 2)(L_{2^k} + 1)\\
&= L_{2^k}^2 - L_{2^k} - 2\\
&= L_{2^k}^2 - L_{2^{k-1}}^2.
\end{aligned}$$

Thus,

$$\frac{F_{2^{k-1}}^2}{L_{2^k}^2 - 1} = \frac{L_{2^k}^2 - L_{2^{k-1}}^2}{5(L_{2^{k-1}}^2 - 1)(L_{2^k}^2 - 1)} = \frac{1}{5}\left(\frac{1}{L_{2^{k-1}}^2 - 1} - \frac{1}{L_{2^k}^2 - 1}\right),$$

which is again telescoping. Consequently, similar to (i), we find

$$\sum_{k=1}^{\infty}\frac{F_{2^{k-1}}^2}{L_{2^k}^2 - 1} = \frac{F_1^2}{L_2^2 - 1} + \frac{1}{5}\cdot\frac{1}{L_2^2 - 1} = \frac{3}{20}.$$

Also solved by Kenneth B. Davenport, Dmitry Fleischman, G. C. Greubel, Russell J. Hendel, Ángel Plaza, and the proposer.

We would like to acknowledge belatedly Adnan Ali for solving B-1146, Dmitry Fleischman for solving B-1142, and Ángel Plaza for solving B-1132.