ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2014. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1146 Proposed by Titu Zvonaru, Comăneşti, Romania.

Prove that $F_{n+2}^2 \geq 5F_{n-1}^2$ for all integers $n \geq 1$.

B-1147 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Find a closed form expression for

$$\sum_{\substack{0\le a,b,c\le n\\ a+b+c=n}} \frac{F_aF_bF_c}{a!b!c!}.$$

B-1148 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Prove that

$$\sum_{k=1}^{\infty} \frac{F_{2^{k-1}}}{L_{2^k}+1} = \frac{1}{\sqrt{5}}.$$

B-1149 **Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.**

For any positive integer number k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recurrently by $u_{n+1} = ku_n + u_{n-1}$ for $n \ge 1$, with respective initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$ and $L_{k,0} = 2$; $L_{k,1} = k$. Prove that

$$\sum_{i=1}^{n-1}(F_{k,i} - \sqrt{F_{k,i}F_{k,i+1}} + F_{k,i+1})^2 + (F_{k,n} - \sqrt{F_{k,n}F_{k,1}} + F_{k,1})^2 \ge \frac{F_{k,n}F_{k,n+1}}{k}, \tag{1}$$

$$\sum_{i=1}^{n-1}(L_{k,i} - \sqrt{L_{k,i}L_{k,i+1}} + L_{k,i+1})^2 + (L_{k,n} - \sqrt{L_{k,n}L_{k,1}} + L_{k,1})^2 \ge \frac{L_{k,n}L_{k,n+1}}{k} - 2, \tag{2}$$

for any positive integer n.

B-1150 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Let n be a positive integer. For any positive integers $r_1, r_2, \ldots, r_n$, find the maximum value of

$$\frac{L_{r_1+r_2+\cdots+r_n}}{F_{r_1}F_{r_2}\cdots F_{r_n}}$$

as a function of n.

SOLUTIONS

*N*th Root of a Product

B-1126 **Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.**
(Vol. 51.2, May 2013)

Let n be a positive integer. Prove that

$$\frac{1}{n}\sqrt[n]{\prod_{i=1}^{n}\left(1+\frac{F_nF_{n+1}}{F_i^2}\right)} \ge 1 + \sqrt[n]{\prod_{i=1}^{n}\frac{F_i^2}{F_nF_{n+1}}}.$$

Solution by Robinson Higuita, Universidad de Antioquia, Colombia.

We know that $\sum_{i=1}^n F_i^2 = F_nF_{n+1}$. This and the AM-GM inequality imply that

$$\begin{aligned}\frac{1}{n}\sqrt[n]{\prod_{i=1}^n\left(1+\frac{F_nF_{n+1}}{F_i^2}\right)} &= \frac{1}{\sqrt[n]{\prod_{i=1}^n F_i^2}}\left(\frac{\sqrt[n]{\prod_{i=1}^n(F_i^2+F_nF_{n+1})}}{n}\right)\\ &\geq \frac{n}{\sum_{i=1}^n F_i^2}\left(\frac{\sqrt[n]{\prod_{i=1}^n(F_i^2+F_nF_{n+1})}}{n}\right)\\ &\geq \frac{\sqrt[n]{\prod_{i=1}^n(F_i^2+F_nF_{n+1})}}{\sqrt[n]{\prod_{i=1}^n F_nF_{n+1}}}\\ &\geq \sqrt[n]{\prod_{i=1}^n\left(\frac{F_i^2}{F_nF_{n+1}}+1\right)}. \end{aligned} \tag{1}$$

Let $c_i = \frac{F_i^2}{F_nF_{n+1}}$, therefore, by AM-GM inequality we have that

$$\sqrt[n]{\frac{1}{\prod_{i=1}^n(c_i+1)}} + \sqrt[n]{\frac{\prod_{i=1}^n c_i}{\prod_{i=1}^n(c_i+1)}} \leq \frac{\sum_{i=1}^n \frac{1}{c_i+1} + \sum_{i=1}^n \frac{c_i}{c_i+1}}{n} \leq 1.$$

Thus,

$$\sqrt[n]{\prod_{i=1}^n\left(\frac{F_i^2}{F_nF_{n+1}}+1\right)} \geq 1 + \sqrt[n]{\prod_{i=1}^n \frac{F_i^2}{F_nF_{n+1}}}. \tag{2}$$

Therefore, the proof follows from (1) and (2).

Remark. A generalization of (2) can be found in [1, p. 17].

References

[1] W. J. Kaczor, *Problems in Mathematical Analysis I*, AMS, 2000.

Also solved by Dmitry Fleischman, Natalie Hilbert and Michael Kubicek (jointly) (students), Ángel Plaza, and the proposer.

A Part of a Bigger Problem!

B-1127 Proposed by George A. Hisert, Berkeley, California. (Vol. 51.2, May 2013)

Prove that, for any integer $n > 2$,

$$-17F_{n-2}^4 + 57F_{n-1}^4 + 402F_n^4 + 113F_{n+1}^4 - 25F_{n+2}^4 = 2F_{n-3}^2L_{n+3}^2 \tag{1}$$

and

$$-17L_{n-2}^4 + 57L_{n-1}^4 + 402L_n^4 + 113L_{n+1}^4 - 25L_{n+2}^4 = 50L_{n-3}^2F_{n+3}^2. \tag{2}$$

Solution by Cheng Lien Lang, Shou University, and Mong Lung Lang, Republic of Singapore (jointly).

Let $X(n)$ be a product of four recurrences, each of which satisfies the recurrence $x_n = x_{n-1} + X_{n-2}$. Applying Jarden's Theorem [1], $X(n)$ satisfies the recurrence

$$X(n+5) = 5X(n+4) + 15X(n+3) - 15X(n+2) - 5X(n+1) + X(n). \tag{3}$$

Let k be a constant and let $X(n)$, $Y(n)$ be sequences that satisfy (3) of the above. It is easy to show that $k(X(n)$ and $X(n) \pm Y(n)$ satisfy (3) as well.

Denote by $A(n)$ and $B(n)$ the left- and right-hand side of (1), respectively. Applying the above results, both $A(n)$ and $B(n)$ satisfy the recurrence (3). Hence, $A(n) = B(n)$ for all n if and only if $A(n) = B(n)$ for $n = 0, 1, 2, 3, 4$ which can be verified easily. Equation (2) can be proved similarly.

The technique can be used to prove various identities [2].

References

[1] D. Jarden, *Recurring Sequences*, 2nd ed., Jerusalem, Riveon Lematematika, 1966.

[2] C. L. Lang and M. L. Lang, *Generalized Binomial Coefficients and Jarden's Theorem*, preprint, `arXiv:math/1305.2146v2 [math.NT]`, 2013.

Also solved by Kenneth B. Davenport, Russell Jay Hendel, Ángel Plaza, and the proposer.

<u>It Could Be A Rational Bound</u>

<u>B-1128</u> Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 51.2, May 2013)

Let n be a positive integer. Prove that

$$\left(\frac{F_{n+2}^2}{1+F_{n+1}^2}\right)\left(\frac{1-F_n^{-2}-F_{n+1}^{-2}}{F_n^2}\right) < \frac{\sqrt{3}}{4}.$$

Solution by Brian D. Beasley, Presbyterian College, SC.

For each positive integer n, we let

$$g(n) = \left(\frac{F_{n+2}^2}{1+F_{n+1}^2}\right)\left(\frac{1-F_n^{-2}-F_{n+1}^{-2}}{F_n^2}\right)$$

and show that $g(n) \le g(3) = 115/288 < \sqrt{3}/4$.

We have

$$g(n) = \frac{(F_{n+1}+F_n)^2(F_n^2F_{n+1}^2 - F_{n+1}^2 - F_n^2)}{F_n^4F_{n+1}^2(1+F_{n+1}^2)} < \frac{(2F_{n+1})^2F_n^2F_{n+1}^2}{F_n^4F_{n+1}^4} = \frac{4}{F_n^2}.$$

Hence, for $n \ge 5$, we obtain $g(n) < 4/F_n^2 \le 4/F_5^2 = 4/25$. We verify that $g(1) = -2$, $g(2) = -9/20$, $g(3) = 115/288$, and $g(4) = 6112/26325$, thus completing the proof.

Also solved by Dmitry Fleischman, Russell Jay Hendel, Robinson Higuita and Bilson Castro (jointly), Natalie Hilbert and Michael Kubicek (jointly, students), Michael Lehotsky (student), Ángel Plaza, David Stone and John Hawkins (jointly), and the proposer.

Inequalities Generalized

B-1129 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.
(Vol. 51.2, May 2013)

Prove that

$$2(L_{n+2}-3)^2 \le (5F_{2n+1}-4)n \tag{1}$$

and

$$2(F_{n+2}-1)^2 \le nF_{2n+1} \tag{2}$$

for any positive integer n.

Solution by Natalie Hilbert and Michael Kubicek (jointly) students at California University of Pennsylvania (CALURMA), California, PA.

Proof. We prove a more general result, namely

$$2(G_{n+2}-b)^2 \le [(3a-b)G_{2n+1}-\mu F_{2n+1}+2a(a-b)]n, \tag{3}$$

where $\{G_n\}_{n\in\mathbb{N}}$ is the generalized Fibonacci sequence with $G_1=a$, $G_2=b$, and $\mu=a^2+ab-b^2$. Using Cauchy-Schwarz inequality [2], $\sum_{i=1}^n G_i = G_{n+2}-b$ [1, Example 11, p. 113], and $\sum_{i=1}^n G_i^2 = G_nG_{n+1}+a(a-b)$ [1, Example 14, p. 113], it is easy to see that

$$\begin{aligned}
2(G_{n+2}-b)^2 = 2\left(\sum_{i=1}^n G_i\right)^2 &\le 2n\sum_{i=1}^n G_i^2\\
&= 2n[G_nG_{n+1}+a(a-b)] = n[2G_nG_{n+1}+2a(a-b)]\\
&\le n[G_n^2+G_{n+1}^2+2a(a-b)].
\end{aligned} \tag{4}$$

The equality $G_n^2+G_{n+1}^2=(3a-b)G_{2n+1}-\mu F_{2n+1}$, see [1, Example 25, p. 113], and (4) prove inequality (3). Inequalities (1) and (2) are obtained from (3) by setting $a=1$, $b=3$, and $a=b=1$, respectively. □

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

[2] WolframMathWorld, *Cauchy-Schwarz*, November 7, 2013, `http://mathworld.wolfram.com/CauchysInequality.html`.

Also solved by Dmitry Fleischman, Russell Jay Hendel, Robinson Higuita, Ángel Plaza, and the proposer.

Almost Déjà vu!

B-1130 Proposed by D. M. Bătineţu–Giurgiu, Neculai Stanciu, and Gabriel Tica, Romania.
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Prove that

$$\sum_{k=1}^{n}\frac{F_k^{2m+2}}{k^{3m}} \geq \frac{4^m F_n^{m+1}F_{n+1}^{m+1}}{n^{2m}(n+1)^{2m}}$$

for all positive real numbers m.

Solution by Rachel Graves, The Military College of South Carolina, SC and Kenneth B. Davenport, Dallas, PA (separately).

We solve this problem similar to Paul S. Bruckman's solution to problem B-1108. It is well-known that $\sum_{k=1}^{n}k^3 = \frac{n^2(n+1)^2}{4}$ and $\sum_{k=1}^{n}F_k^2 = F_nF_{n+1}$. We use these identities and Hölder's Inequality to prove that the proposed inequality is true.

Hölder's Inequality states that if $\{a_k\}_{k\geq 1}$ and $\{b_k\}_{k\geq 1}$ are non-negative sequences, and for all $p>0$ and $q>0$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $n\geq 1$, then

$$\sum_{k=1}^{n}a_kb_k \leq \left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}b_k^q\right)^{\frac{1}{q}}.$$

Let $a_k=(k^3)^{\frac{1}{p}}$, $b_k=\frac{F_k^3}{(k^3)^{\frac{1}{p}}}$. Substituting in Hölder's Inequality, we obtain

$$\sum_{k=1}^{n}(k^3)^{\frac{1}{p}}\left(\frac{F_k^2}{(k^3)^{\frac{1}{p}}}\right) \leq \left(\sum_{k=1}^{n}\left((k^3)^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}\left(\frac{F_k^2}{(k^3)^{\frac{1}{p}}}\right)^q\right)^{\frac{1}{q}},$$

which simplifies to

$$\sum_{k=1}F_k^2 \leq \left(\sum_{k=1}^{n}k^3\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}\frac{F_k^{2q}}{(k^3)^{\frac{q}{p}}}\right)^{\frac{1}{q}}.$$

Raising both sides of the inequality to the power of q, we have

$$\left(\sum_{k=1}^{n}F_k^2\right)^q \leq \left(\sum_{k=1}^{n}k^3\right)^{\frac{q}{p}}\left(\sum_{k=1}^{n}\frac{F_k^{2q}}{(k^3)^{\frac{q}{p}}}\right).$$

We now can see that we should let $q=m+1$ and it follows that $p=\frac{m}{m+1}$. Substituting these values gives

$$(F_nF_{n+1})^{m+1} \leq \left(\frac{n(n+1)}{2}\right)^{2m}\left(\sum_{k=1}^{n}\frac{F_k^{2m+2}}{k^{3m}}\right).$$

This inequality can easily be rewritten as

$$\frac{4^m(F_nF_{n+1})^{m+1}}{n^{2m}(n+1)^{2m}} \leq \sum_{k=1}^{n}\frac{F_k^{2m+2}}{k^{3m}}.$$

Also solved by Dmitry Fleischman, Robinson Higuita and Bilson Castro (jointly), Ángel Plaza, and the proposer.

We would like to acknowledge the solutions to problems B-1116 and B-1120 by Anastasios Kotronis, B-1121 by Kenneth Davenport, and B-1124 by Rattanapol Wasutharat.