# ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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*Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.*

*Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2023. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."*

*The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.*

## BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = \big(1+\sqrt{5}\,\big)/2$, $\beta = \big(1-\sqrt{5}\,\big)/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

## PROBLEMS PROPOSED IN THIS ISSUE

**B-1313** **(Corrected) Proposed by Daniel Văcaru, Economical College Maria Teiuleanu, Piteşti, Romania, and Mihály Bencze, Aprily Lajos, Braşov, Romania.**

For $a \le -1$, show that

$$\sum_{k=1}^{n} F_k(F_{n+2} - F_k - 1)^a \ge \left(\frac{n-1}{n}\right)^a (F_{n+2}-1)^{a+1},$$

$$\sum_{k=1}^{n} F_k^2(F_nF_{n+1} - F_k^2)^a \ge \left(\frac{n-1}{n}\right)^a (F_nF_{n+1})^{a+1}.$$

**B-1316** **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Prove that

$$\sum_{n=0}^{\infty} \tanh^{-1}\frac{1}{L_{2\cdot 3^n}} = \frac{1}{4}\ln 5.$$

**B-1317** **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For every integer $n$, find the real roots of the equation

$$x^4 - 2F_{n+1}^2 x^2 - 4F_{n+1}F_{n-1}^2 x + F_{n+1}^4 - F_{n-1}^4 = 0.$$

**B-1318** **Proposed by Kenny B. Davenport, Dallas, PA.**

Prove that

$$\sum_{k=1}^{n}(-1)^k F_{k-1}F_kF_{k+1} = \frac{(-1)^n}{12}\left(-4F_n^3 - F_{n+1}^3 + 2F_{n+2}^3 + 4F_{n+3}^3 - F_{n+4}^3\right) - \frac{1}{2}.$$

**B-1319** **Proposed by Toyesh Prakash Sharma (student), St. C. F. Andrews School, Agra, India.**

Show that, for $n \geq 4$,

$$F_n^{\frac{1}{F_n}} L_n^{\frac{1}{L_n}} \geq \left(F_{n+1}^2\right)^{\frac{1}{F_{n+1}}}.$$

**B-1320** **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Prove that

$$\sum_{n=1}^{\infty}\frac{4}{(F_nF_{n+3})^2} = \sum_{n=1}^{\infty}\frac{1}{(F_nF_{n+2})^2} + \frac{1}{4}.$$

## SOLUTIONS

### An Infinite Product with Subscripts That Are Multiples of Powers of Two

**B-1296** **Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 59.4, November 2021)**

For integers $s > r \geq 0$, evaluate

$$\prod_{n=1}^{\infty}\left(1 + \frac{L_{2^n r}}{L_{2^n s}}\right).$$

**Solution by Michel Bataille, Rouen, France.**

Let $P(r,s)$ denote the product to be evaluated. We claim that $P(r,s) = \dfrac{\sqrt{5}\,F_{2s}}{L_{2s} - L_{2r}}$. For any positive integer $N$, let

$$P_N(r,s) = \prod_{n=1}^{N}\left(1 + \frac{L_{2^n r}}{L_{2^n s}}\right) = \frac{\prod_{n=1}^{N}(L_{2^n s} + L_{2^n r})}{\prod_{n=1}^{N} L_{2^n s}}.$$

We first prove that

$$\prod_{n=1}^{N} L_{2^n s} = \frac{F_{2^{N+1} s}}{F_{2s}} \tag{1}$$

The proof is by induction. Because of the general formula $F_m L_m = F_{2m}$, we have $L_{2s} = \frac{F_{4s}}{F_{2s}}$ and (1) holds for $N = 1$. Assuming that (1) holds for some integer $N \geq 1$, we obtain

$$\prod_{n=1}^{N+1} L_{2^n s} = \left(\frac{F_{2^{N+1}s}}{F_{2s}}\right) \cdot L_{2^{N+1}s} = \frac{F_{2\cdot 2^{N+1}s}}{F_{2s}} = \frac{F_{2^{N+2}s}}{F_{2s}}.$$

This completes the inductive step and the proof of identity (1).

For integers $m$ and $p$, it is easy to derive from the Binet formulas that

$$L_{m-p} \cdot L_{m+p} = L_{2m} + (-1)^{m-p} L_{2p}.$$

We deduce that, if $n > 1$,

$$L_{2^{n-1}(s-r)} \cdot L_{2^{n-1}(s+r)} = L_{2^n s} + L_{2^n r},$$

and it follows that, for $N \geq 2$,

$$\begin{aligned}\prod_{n=1}^{N} (L_{2^n s} + L_{2^n r}) &= (L_{2s} + L_{2r}) \left(\prod_{n=2}^{N} L_{2^{n-1}(s-r)}\right) \left(\prod_{n=2}^{N} L_{2^{n-1}(s+r)}\right) \\ &= (L_{2s} + L_{2r}) \frac{F_{2^N(s-r)} F_{2^N(s+r)}}{F_{2(s-r)} F_{2(s+r)}}.\end{aligned}$$

This result and (1) lead to

$$P_N(r,s) = \frac{F_{2s}(L_{2s} + L_{2r})}{F_{2(s-r)} F_{2(s+r)}} \cdot \frac{F_{2^N(s-r)} F_{2^N(s+r)}}{F_{2^{N+1}s}}.$$

Using Binet's formulas, we readily obtain

$$5F_{2(s-r)} F_{2(s+r)} = L_{4s} - L_{4r} = (L_{2s} + L_{2r})(L_{2s} - L_{2r}).$$

In addition, since $F_m \sim \frac{\alpha^m}{\sqrt{5}}$ as $m \to \infty$, we have $\lim_{N\to\infty} \frac{F_{2^N(s-r)} F_{2^N(s+r)}}{F_{2^{N+1}s}} = \frac{1}{\sqrt{5}}$. Finally,

$$P(r,s) = \lim_{N\to\infty} P_N(r,s) = \frac{5F_{2s}}{L_{2s} - L_{2r}} \cdot \frac{1}{\sqrt{5}} = \frac{\sqrt{5}\,F_{2s}}{L_{2s} - L_{2r}},$$

as claimed.

**Also solved by Luis Gerardo Hernández Chávez (undergraduate), Steve Edwards, Dmitry Fleischman, Robet Frontczak, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, David Terr, Andrés Ventas, and the proposer.**

## Two Special Cases of a More General Inequality

**B-1297 Proposed by D. M. Bătineţu-Giurgui, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**
**(Vol. 59.4, November 2021)**

For integers $m > 1$ and $n \geq 1$, prove that

(A) $\displaystyle\sum_{k=1}^{n} (1 + F_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} F_n F_{n+1}$

(B) $\displaystyle\sum_{k=1}^{n} (1 + L_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} \big(L_n L_{n+1} - 2\big)$

**Solution 1 by Hideyuki Ohtsuka, Saitama, Japan.**

The inequality holds for any real number $m \geq 1$. Let $m \geq 1$. For $x > 0$, define $f(x) = \frac{(x+1)^{m+1}}{x^m}$. Then, we have $f'(x) = \frac{(x+1)^m(x-m)}{x^{m+1}}$. Since $f'(x) < 0$ for $0 < x < m$, $f'(x) > 0$ for $x > m$, and $f'(m) = 0$, we see that

$$f(x) = \frac{(x+1)^{m+1}}{x^m} \geq f(m) = \frac{(m+1)^{m+1}}{m^m}.$$

From the above inequality, we have

$$(1+x)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} x^{2m} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} x^2.$$

Therefore, we obtain

$$\sum_{k=1}^{n}(1+F_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} \sum_{k=1}^{n} F_k^2 = \frac{(m+1)^{2(m+1)}}{m^{2m}} F_n F_{n+1},$$

and

$$\sum_{k=1}^{n}(1+L_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} \sum_{k=1}^{n} L_k^2 = \frac{(m+1)^{2(m+1)}}{m^{2m}} (L_n L_{n+1} - 2).$$

**Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA.**

Starting with

$$\frac{m(1+F_k)}{m+1} = \frac{\overbrace{1+1+\cdots+1}^{m} + mF_k}{m+1} \geq \sqrt[m+1]{mF_k},$$

which follows from the AM-GM inequality, we obtain

$$\sum_{k=1}^{n}\left(\frac{m(1+F_k)}{m+1}\right)^{2(m+1)} \geq \sum_{k=1}^{n} m^2 F_k^2 = m^2 F_n F_{n+1}.$$

Multiplying both sides by $\left(\frac{m+1}{m}\right)^{2(m+1)}$ yields the desired inequality. The proof of (B) uses a similar argument.

*Editor's Notes*: It is clear from the proofs, as Plaza also observed, that the inequality holds for any positive real numbers $x_k$:

$$\sum_{k=1}^{n}(1+x_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} \sum_{k=1}^{n} x_k^2.$$

**Also solved by Michel Bataille, Dmitry Fleischman, Ángel Plaza, Albert Stadler, Andrés Ventas, Dan Weiner, and the proposer.**

## Write It As a Telescopic Sum

**B-1298 Proposed by Diego Rattaggi, Realgymnasium Rämibühl, Zürich, Switzerland.**
**(Vol. 59.4, November 2021)**

For any positive integer $k$, prove that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2kn+k+1}^2 + F_k^2} = \frac{1}{\alpha F_{2k}}.$$

**Solution by Raphael Schumacher (graduate student, teaching diploma mathematics), ETH Zurich, Switzerland.**

We have by using Catalan's identity $F_a^2 - (-1)^{a+b} F_b^2 = F_{a+b}F_{a-b}$ and d'Ocagne's identity $F_c F_{d+1} - F_d F_{c+1} = (-1)^d F_{c-d}$ that, for all integers $n \geq 0$ and $k \geq 1$,

$$\begin{aligned} \frac{1}{F_{2kn+k+1}^2 + F_k^2} &= \frac{F_{2k}}{F_{2k}F_{2kn+2k+1}F_{2kn+1}} \\ &= \frac{F_{2kn+2k}F_{2kn+1} - F_{2kn}F_{2kn+2k+1}}{F_{2k}F_{2kn+2k+1}F_{2kn+1}} \\ &= \frac{F_{2kn+2k}}{F_{2k}F_{2kn+2k+1}} - \frac{F_{2kn}}{F_{2k}F_{2kn+1}}. \end{aligned}$$

By telescoping, we deduce that

$$\sum_{n=0}^{m} \frac{1}{F_{2kn+k+1}^2 + F_k^2} = \sum_{n=0}^{m} \left( \frac{F_{2kn+2k}}{F_{2k}F_{2kn+2k+1}} - \frac{F_{2kn}}{F_{2k}F_{2kn+1}} \right) = \frac{F_{2km+2k}}{F_{2k}F_{2km+2k+1}},$$

which implies that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2kn+k+1}^2 + F_k^2} = \lim_{m\to\infty} \left( \frac{F_{2km+2k}}{F_{2k}F_{2km+2k+1}} \right) = \frac{1}{\alpha F_{2k}},$$

because $\lim_{n\to\infty} \left( \frac{F_n}{F_{n+1}} \right) = \frac{1}{\alpha}$.

*Editor's Notes*: Frontczak obtained the Lucas analog

$$\sum_{n=0}^{\infty} \frac{1}{L_{2kn+k+1}^2 - 5F_k^2} = \frac{L_{2k-1} + \alpha^{-2k}}{5F_{2k}^2}, \qquad k \geq 1.$$

Ventas remarked that a generalization can be found in [1].

### References

[1] B. S. Popov, *Summation of reciprocal series of numerical functions of second order*, The Fibonacci Quarterly, **24.1** (1986), 17–21.

**Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Luis Gerardo Hernández Chávez (undergraduate), Steve Edwards, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Hideyuki Ohtsuka, Ángel Plaza, Jason L. Smith, Albert Stadler, Andrés Ventas, and the proposer.**

## Factorial and Factorial

**B-1299 Proposed by Toyesh Prakash Sharma (high school student), St. C. F. Andrews School, Agra, India.**
**(Vol. 59.4, November 2021)**

Let $n$ be a positive even integer. Prove that

$$\exp\left(\sum_{k=1}^{n} F_{k-1}F_{k+1}\right) > \prod_{k=1}^{n}\left[(F_k)!\,(F_k+1)!\right].$$

**Solution by Michael R. Bacon and Charles K. Cook (jointly), Sumter, SC.**

We shall prove the broader results that

$$\begin{aligned} F_{n-1}F_{n+1} &> \ln\left[(F_n)!\,(F_n+1)!\right], && \text{for any integer } n \geq 2,\\ L_{n-1}L_{n+1} &> \ln\left[(L_n)!\,(L_n+1)!\right], && \text{for } n=1 \text{ and any integer } n \geq 3. \end{aligned}$$

Note that the result fails for the Fibonacci numbers when $n = 1$ and fails for the Lucas numbers when $n = 2$. These two general results imply that

$$\begin{aligned} \exp\left(\sum_{k=2}^{n} F_{k-1}F_{k+1}\right) &> \prod_{k=2}^{n}\left[(F_k)!\,(F_k+1)!\right], && \text{for any integer } n \geq 2,\\ \exp\left(\sum_{k=3}^{n} L_{k-1}L_{k+1}\right) &> \prod_{k=3}^{n}\left[(L_k)!\,(L_k+1)!\right], && \text{for any integer } n \geq 3. \end{aligned}$$

They, with the numeric values for the special cases of $k = 2, 3$, imply that

$$\begin{aligned} \exp\left(\sum_{k=2}^{n} F_{k-1}F_{k+1}\right) &> \prod_{k=1}^{n}\left[(F_k)!\,(F_k+1)!\right], && \text{for } \textit{any} \text{ integer } n \geq 2,\\ \exp\left(\sum_{k=3}^{n} L_{k-1}L_{k+1}\right) &> \prod_{k=1}^{n}\left[(L_k)!\,(L_k+1)!\right], && \text{for } \textit{any} \text{ integer } n \geq 1. \end{aligned}$$

We now prove the broader results stated in the beginning of the solution. The general proof uses the inequalities

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n \geq F_n^2 - 1,$$
$$L_{n-1}L_{n+1} = L_n^2 - 5\,(-1)^n \geq L_n^2 - 5,$$

and an upper bound for Stirling's approximation of $\ln(k!)$ that can be found in [1] or [2]:

$$\frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln k + k\ln k - k + \frac{1}{12k} > \ln(k!).$$

Since $\ln\left[k!\,(k+1)!\right] = 2\ln(k!) + \ln(k+1)$, we have

$$\ln(2\pi) + \ln k + 2k\ln k - 2k + \frac{1}{6k} + \ln(k+1) > \ln\left[(k!\,(k+1)!\right].$$

Define, for $x \geq 1$,

$$\begin{aligned} f(x) &= x^2 - 1 - \ln(2\pi) - \ln(x) - 2x\ln(x) + 2x - \frac{1}{6x} - \ln(x+1),\\ g(x) &= x^2 - 5 - \ln(2\pi) - \ln(x) - 2x\ln(x) + 2x - \frac{1}{6x} - \ln(x+1). \end{aligned}$$

We find
$$f'(x) = g'(x) = 2(x - \ln(x)) - \left(\frac{1}{x} + \frac{1}{x+1}\right) + \frac{1}{6x^2}.$$
When $x \geq 1$, the smallest value of $x - \ln(x)$ is $1 - \ln(1) = 1$, the largest value of $\frac{1}{x} + \frac{1}{x+1}$ is $\frac{1}{1} + \frac{1}{2} = \frac{3}{2}$, and the smallest value of $\frac{1}{6x^2}$ is $\frac{1}{6}$. So $g'(x) = f'(x) \geq 2 \cdot 1 - \frac{3}{2} + \frac{1}{6} > 0$. Both $f$ and $g$ are strictly increasing functions when $x \geq 1$. The smallest integer value for which $f(x) > 0$ is $x = 2$; so when $x \geq 2$,
$$x^2 - 1 > \ln(2\pi) + \ln(x) + 2x\ln(x) - 2x + \frac{1}{6x} + \ln(x+1).$$
The smallest value of $x$ for which $g(x) > 0$ is $x = 4$; hence when $x \geq 4$,
$$x^2 - 5 > \ln(2\pi) + \ln(x) + 2x\ln(x) - 2x + \frac{1}{6x} + \ln(x+1).$$
We conclude that the smallest Fibonacci and Lucas numbers for which the inequality holds are $F_3 = 2$ and $L_3 = 4$, respectively. Therefore, for $n \geq 3$,
$$\begin{aligned} F_{n-1}F_{n+1} &\geq F_n^2 - 1 \\ &> \ln(2\pi) + \ln(F_n) + 2F_n\ln(F_n) - 2F_n + \frac{1}{6F_n} + \ln(F_n + 1) \\ &> 2\ln\left[(F_n)!\right] + \ln(F_n + 1) \\ &= \ln\left[(F_n)!(F_n + 1)!\right], \end{aligned}$$
and similarly
$$L_{n-1}L_{n+1} \geq L_n^2 - 5 > \ln\left[(L_n)!(L_n + 1)!\right].$$
This establishes the boarder results and thus completes the solution.

## References


[1] H. Robbins, *A remark of Stirling's formula*, Amer. Math. Monthly, **62** (1955), 26–29.

[2] E. W. Weisstein, Stirling's Approximation. *MathWorld — A Wolfram Web Resource*, `https://mathworld.wolfram.com/StirlingsApproximation.html`


**Also solved by Thomas Achammer, Michel Bataille, Dmitry Fleischman, Luke Paluso (graduate student), Albert Stadler, Andrés Bentas, and the proposer.**

## Millin Series

**B-1300 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.**
**(Vol. 59.4, November 2021)**

Let the sequence $\{a_n\}_{n\geq 0}$ be defined by $a_0 = 1$, $a_1 = 3$, and $a_{n+2} = a_{n+1}\big(5a_n^2 + 2\big)$. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a_n}$.

**Composite Solution by Brian Bradie, Christopher Newport University, Newport News, VA, and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.**

Note that
$$a_0 = 1 = F_2, \qquad \text{and} \qquad a_1 = 3 = F_4.$$

Assume $a_k = F_{2^{k+1}}$ for $k = 0, 1, \ldots, n$ for some positive integer $n$. Then,

$$a_{n+1} = a_n\left(5a_{n-1}^2 + 2\right) = F_{2^{n+1}}\left(5F_{2^n}^2 + 2\right) = F_{2^{n+1}}L_{2^{n+1}} = F_{2^{n+2}}.$$

Thus, $a_n = F_{2^{n+1}}$ for all integers $n \geq 0$ by induction. Then,

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{F_{2^n}}.$$

Since

$$\frac{1}{F_{2^n}} = \frac{\sqrt{5}}{\alpha^{2^n} - \beta^{2^n}} = \frac{\sqrt{5}\,\alpha^{2^n}}{\alpha^{2^{n+1}} - 1} = \sqrt{5}\left(\frac{1}{\alpha^{2^n} - 1} - \frac{1}{\alpha^{2^{n+1}} - 1}\right),$$

the proposed sum telescopes. We find

$$\sum_{n}^{\infty} \frac{1}{a_n} = \sqrt{5} \lim_{n\to\infty}\left(\frac{1}{\alpha^2 - 1} - \frac{1}{\alpha^{2^{n+1}} - 1}\right) = \frac{\sqrt{5}}{\alpha^2 - 1} = \frac{\sqrt{5}}{\alpha}.$$

The series $\sum_{n=0}^{\infty} 1/F_{2^n}$ is known as the Millin series, whose value is $\frac{1}{2}\left(7 - \sqrt{5}\right)$ [1]. Evaluation of this series is the subject of a problem proposed by Millin [3], with a solution provided by Shannon [4]. Hoggart and Bicknell [2] described 10 additional techniques for determining the value of this series.

## References


[1] I. J. Good, *A reciprocal series of Fibonacci numbers*, The Fibonacci Quarterly, **12.4** (1974), 346.

[2] V. E. Hoggart and M. Bicknell, *A primer for the Fibonacci numbers, Part XV: Variations on summing a series of reciprocals of Fibonacci numbers*, The Fibonacci Quarterly, **14.3** (1976), 272–278.

[3] D. A. Millin, *Problem H-237*, The Fibonacci Quarterly, **12.3** (1974), 309.

[4] A. G. Shannon, *Solution to Problem H-237*, The Fibonacci Quarterly, **14.2** (1976), 186–187.


**Also solved by Thomas Achammer, Michel Bataille, Luis Gerardo Hernández Chávez (undergraduate), Charles K. Cook and Michael R. Bacon (jointly), Charles K. Cook (alternate solution), Steve Edwards, Dmitry Fleischman, G. C. Greubel, Hideyuki Ohtsuka, Luke Paluso (graduate student), Raphael Schumacher (graduate student), Albert Stadler, Seán M. Stewart, David Terr, Andrés Ventas, and the proposer.**