ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2021. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-51 Proposed by Douglas Lind, Falls Church, VA.
(Vol. 2.3, October 1964)

Let $\phi(n)$ be the Euler totient and let $\phi^k(n)$ be defined by $\phi^1(n) = \phi(n)$, $\phi^{k+1}(n) = \phi(\phi^k(n))$. Prove that $\phi^n(F_n) = 1$, where F_n is the nth Fibonacci number.

Editor's Note: This is the last unsolved problem from the old issues. No solutions have appeared, so we feature the problem again, and invite the readers to solve it.

B-1281 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all positive integers n and m, prove that

$$L_1 + \sqrt[n]{\sum_{k=1}^{m} L_k^n} \le \sqrt[n]{\sum_{k=1}^{m} F_{k+1}^n} + F_{m+1}.$$

B-1282 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any positive integer n, find closed form expressions for the sums

$$\sum_{k=1}^{n} F_{3k}F_{3k+1}, \qquad \text{and} \qquad \sum_{k=1}^{n} F_{2F_{3k}}F_{2F_{3k+1}}.$$

B-1283 Proposed by Michel Bataille, Rouen, France.

For positive integers m, n, evaluate in closed form:

$$\sum_{j=1}^{n} \binom{2n}{n-j} \frac{F_{mn-4j}+F_{mn+4j}}{F_{mn}}.$$

B-1284 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Gran Canaria, Spain.

Let $(x_n)_{n\geq 0}$ be the sequence recurrently defined by $x_{n+1} = x_n + x_{n-1}$ for $n \geq 1$, with initial conditions $x_0 \geq 0$ and $x_1 \geq 1$. For $n \geq 2$, prove that

$$\ln\left(\frac{1}{n-1}\left(\frac{x_2}{x_1}+\frac{x_3}{x_2}+\cdots+\frac{x_n}{x_{n-1}}\right)\right) \geq \frac{2}{n-1}\left(\frac{x_0}{x_3}+\frac{x_1}{x_4}+\cdots+\frac{x_{n-2}}{x_{n+1}}\right).$$

B-1285 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let $i = \sqrt{-1}$. For any integer $n \geq 0$, prove that

(i) $\displaystyle\sum_{k=-n}^{n} \binom{2n}{n+k}\left(e^{\frac{2k\pi i}{5}} + e^{\frac{4k\pi i}{5}}\right) = L_{2n};$

(ii) $\displaystyle\sum_{k=-n}^{n} \binom{2n}{n+k}\left(e^{\frac{k\pi i}{5}} + (-1)^n e^{\frac{3k\pi i}{5}}\right) = \left(\sqrt{5}\,\right)^n L_n.$

SOLUTIONS

The Third Oldie from the Vault

B-835 Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY.
(Vol. 35.3, August 1997)

In a sequence of coin tosses, a *single* is a term (H or T) that is not the same as any adjacent term. For example, in the sequence HHTHHHTH, the singles are the terms in positions 3, 7, and 8. Let $S(n,r)$ be the number of sequences of n coin tosses that contain exactly r singles. If $n \geq 0$, and p is prime, find the value modulo p of $\frac{1}{2}S(n+p-1,p-1)$.

Editor's Note: Albert Stadler kindly remarked that the problem was republished as Problem B-899 [1], with a solution that appeared in [3]. For a more detailed discussion and related problems, see [2].

References

[1] D. Bloom, *Problem B-899*, The Fibonacci Quarterly, **38.2** (2000), 181.
[2] D. Bloom, *Singles in a sequence of coin tosses*, The College Mathematics Journal, **29.2** (1998), 120–127.
[3] D. Bloom, *Solution to Problem B-899*, The Fibonacci Quarterly, **39.2** (2001), 182–183.

Also solved by Raphael Schumacher (student), and the proposer.

Powers of Three in the Subscripts

B-1261 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.
(Vol. 58.1, February 2020)

Show that $\prod_{n=0}^{\infty} \frac{L_{3^n}^2+1}{L_{3^n}^2+3} = \prod_{n=0}^{\infty} \frac{5F_{3^n}^2-3}{5F_{3^n}^2-1}$, and determine the exact value of the limit.

Solution by Hideyuki Ohtsuka, Saitama, Japan.

For any integer k, we have

$$L_k^2-(-1)^k = 5F_k^2+3(-1)^k = \frac{F_{3k}}{F_k},$$

because, according to Binet's formulas, all three expressions are equal to $\alpha^{2k}+\beta^{2k}+(-1)^k$. We also have

$$L_k^2-3(-1)^k = 5F_k^2+(-1)^k = \frac{L_{3k}}{L_k},$$

because all three expressions are equal to $\alpha^{2k}+\beta^{2k}-(-1)^k$. Using these identities, we obtain

$$\prod_{n=0}^{m} \frac{L_{3^n}^2+1}{L_{3^n}^2+3} = \prod_{n=0}^{m} \frac{5F_{3^n}^2-3}{5F_{3^n}^2-1} = \prod_{n=0}^{m} \frac{F_{3^{n+1}}}{F_{3^n}} \cdot \frac{L_{3^n}}{L_{3^{n+1}}} = \frac{F_{3^{m+1}}}{F_{3^0}} \cdot \frac{L_{3^0}}{L_{3^{m+1}}} = \frac{F_{3^{m+1}}}{L_{3^{m+1}}}.$$

Therefore,

$$\prod_{n=0}^{\infty} \frac{L_{3^n}^2+1}{L_{3^n}^2+3} = \prod_{n=0}^{\infty} \frac{5F_{3^n}^2-3}{5F_{3^n}^2-1} = \lim_{m\to\infty} \frac{F_{3^{m+1}}}{L_{3^{m+1}}} = \frac{1}{\sqrt{5}}.$$

Editor's Note: Several solvers used the product formulas to derive the identities $L_{3^n}^2+1 = L_{2\cdot 3^n}-1$ and $L_{3^n}^2+3 = L_{2\cdot 3^n}+1$, to which they applied the following identities (they can be found in [1], and can be proved by induction) $\prod_{n=0}^{m}(L_{2\cdot 3^n}-1) = F_{3^{m+1}}$ and $\prod_{n=0}^{m}(L_{2\cdot 3^n}+1) = L_{3^{m+1}}$ to finish their proofs.

Reference

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

Also solved by Brian Beasley, Brian Bradie, Charles K. Cook, Dmitry Fleischman, I. V. Fedak, Raphael Schumacher (student), Jason L. Smith, Albert Stadler, David Terr, and the proposer.

Stirling Approximation for Double Factorials

B-1262 Proposed by D. M. Bătineţu-Giurgiu, Mateo Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 58.1, February 2020)

Compute

$$\lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!\,F_{n+1}}-\sqrt[3n]{(2n-1)!!\,F_n}\right)\sqrt[3]{n^2}.$$

Solution by Dmitry Fleischman, Santa Monica, CA.

Because $(2n-1)!!=\frac{(2n)!}{2^n\,n!}$, according to Stirling's approximation,

$$(2n-1)!!=\frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}\left[1+\frac{1}{24n}+O\left(\frac{1}{n^2}\right)\right]}{2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\left[1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right]}=\sqrt{2}\left(\frac{2n}{e}\right)^n\left[1-\frac{1}{24n}+O\left(\frac{1}{n^2}\right)\right].$$

Recall that

$$F_n=\frac{\alpha^n\left[1-\left(\frac{\beta}{\alpha}\right)^n\right]}{\sqrt{5}},$$

where $\left|\frac{\beta}{\alpha}\right|<0.4$, and

$$\left[1-\left(\frac{\beta}{\alpha}\right)^n\right]^{\frac{1}{3n}}=1-\frac{\left(\frac{\beta}{\alpha}\right)^n}{3n}+o\left(\left(\frac{\beta}{\alpha}\right)^{2n}\right).$$

For $a>0$, we find

$$\left(\sqrt{a}\right)^{\frac{1}{3n}}=e^{\frac{\ln(a)}{6n}}=1+\frac{\ln(a)}{6n}+O\left(\frac{1}{n^2}\right).$$

Together with

$$\left[1-\frac{1}{24n}+O\left(\frac{1}{n^2}\right)\right]^{\frac{1}{3n}}=1-\frac{1}{72n^2}+O\left(\frac{1}{n^3}\right),$$

we gather that

$$\left[(2n-1)!!\,F_n\right]^{\frac{1}{3n}}=\left(\frac{2n\alpha}{e}\right)^{\frac{1}{3}}\left[1+O\left(\frac{1}{n}\right)\right].$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!\,F_{n+1}}-\sqrt[3n]{(2n-1)!!\,F_n}\right)\sqrt[3]{n^2}\\
&=\lim_{n\to\infty}\left(\frac{2\alpha}{e}\right)^{\frac{1}{3}}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)\sqrt[3]{n^2}\\
&=\lim_{n\to\infty}\left(\frac{2\alpha}{e}\right)^{\frac{1}{3}}\frac{[(n+1)-n]\sqrt[3]{n^2}}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n+1}\sqrt[3]{n}+\sqrt[3]{n^2}}\\
&=\frac{1}{3}\left(\frac{2\alpha}{e}\right)^{\frac{1}{3}}.
\end{aligned}$$

Editor's Notes: The solutions we received can be divided into two categories, similar to the solutions to two recent problems [1, 2]. Some used the argument in [3], others used Stirling approximation for factorials, as in [4].

References

[1] D. M. Bătineţu-Giurgiu and N. Stanciu, *Problem B-1229*, The Fibonacci Quarterly, **56.2** (2018), 178.

[2] D. M. Bătineţu-Giurgiu, N. Stanciu, and G. Tica, *Problem B-1202*, The Fibonacci Quarterly, **55.1** (2017), 83.

[3] D. M. Bătineţu-Giurgiu, N. Stanciu and G. Tica, *Solution to Problem B-1202*, The Fibonacci Quarterly, **56.1** (2018), 84–85.

[4] D. Terr, *Solution to Problem B-1229*, The Fibonacci Quarterly, **57.2** (2019), 180–181.

Also solved by Brian Bradie, Charles Burnette, I. V. Fedak, G. C. Greubel, Ángel Plaza, Raphael Schumacher (student), and the proposers.

Pairing Up Fibonacci and Lucas with Pell and Pell-Lucas

B-1263 Proposed by Stanley Rabinowitz, Milford, NH.
(Vol. 58.1, February 2020)

Let P_n denote the nth Pell number. Find a recurrence relation for $X_n = F_n + P_n$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA.

Let
$$X_n = F_{k_1,n} + F_{k_2,n},$$
where $F_{k,n}$ denotes the k-Fibonacci numbers defined as $F_{k,0} = 0$, $F_{k,1} = 1$, and
$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \qquad n \geq 1.$$
It is a routine exercise to verify that $\frac{t}{1-kt-t^2}$ is the generating function for $F_{k,n}$. So, the generating function for X_n is
$$\begin{aligned}\mathcal{X}(t) = \sum_{n=0}^{\infty} X_n t^n &= \frac{t}{1-k_1\,t-t^2} + \frac{t}{1-k_2\,t-t^2}\\ &= \frac{2t-(k_1+k_2)\,t^2-2t^3}{1-(k_1+k_2)\,t+(k_1k_2-2)\,t^2+(k_1+k_2)\,t^3+t^4}.\end{aligned}$$
Because $\left[1-(k_1+k_2)\,t+(k_1k_2-2)\,t^2+(k_1+k_2)\,t^3+t^4\right]\mathcal{X}(t)$ is a cubic polynomial, the coefficient of t^n must be zero for $n \geq 4$. This leads to the recurrence relation
$$X_{n+1} = (k_1+k_2)X_n - (k_1k_2-2)X_{n-1} - (k_1+k_2)X_{n-2} - X_{n-3}, \qquad n \geq 3.$$
Letting $k_1 = 1$ and $k_2 = 2$ yields $F_{k_1,n} = F_n$ and $F_{k_2,n} = P_n$, and $X_n = F_n + P_n$ satisfies the recurrence relation
$$X_{n+1} = 3X_n - 3X_{n-2} - X_{n-3}, \qquad n \geq 3.$$
Notice that the same recurrence relation also applies to $X_n = L_n + P_n$, $X_n = F_n + Q_n$, and $X_n = L_n + Q_n$, where Q_n denotes the Pell-Lucas number.

Solution 2 by Jason L. Smith, Richland Community College, Decatur, IL.

Define the operator D on a sequence $\{a_n\}$ such that $Da_n = a_{n-1}$. From this definition, we can see that $D^k a_n = a_{n-k}$. It can also be shown that polynomials in D having constant coefficients commute; that is, $p(D)q(D)a_n = q(D)p(D)a_n$ for any polynomials p and q whose coefficients do not depend on n. Using this notation, we can express the Fibonacci and Pell recursions as $\left(1 - D - D^2\right)F_n = 0$ and $\left(1 - 2D - D^2\right)P_n = 0$, respectively. Let us say that each expression in D "annihilates" its respective sequence. From this, it can be seen that the operator

$$\left(1 - D - D^2\right)\left(1 - 2D - D^2\right) = 1 - 3D + 3D^3 + D^4$$

annihilates $X_n = F_n + P_n$. From this, we recover the recursion $X_n - 3X_{n-1} + 3X_{n-3} + X_{n-4} = 0$, or $X_n = 3X_{n-1} - 3X_{n-3} - X_{n-4}$.

Also solved by Brian Beasley, Charles Burnette, Charles K. Cook, Kenny B. Davenport, Pridon Davlianidze, Tom Edgar, I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Kabil Kumar Gurjar, Russell Jay Hendel, Carl Libis, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (student), Jason L. Smith (second solution), Albert Stadler, and the proposer.

It's All About Catalan

B-1264 Pridon Davlianidze, Tbilisi, Republic of Georgia. (Vol. 58.1, February 2020)

Prove that

(A) $\displaystyle\prod_{n=2}^{\infty}\left(1 + \frac{1}{F_{2n-1}^2}\right) = \frac{\alpha^2}{2}$

(B) $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{1}{F_{2n}^2}\right) = \frac{\alpha^2}{3}$

(C) $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{1}{F_{2n-1}^2}\right)\left(1 + \frac{1}{F_{2n}^2}\right) = \frac{\alpha}{2}$

Solution by Anlly Daniela Giraldo Melo and Alejandro Cardona Castrillón (both students) (jointly), Universidad de Antioquia, Medellín Antioquia, Colombia.

Using the Catalan's formula $F_{m+k}F_{m-k} - F_m^2 = (-1)^{m+k+1}F_k^2$, for part (A), we obtain

$$\prod_{n=2}^{\infty}\left(1 + \frac{1}{F_{2n-1}^2}\right) = \prod_{n=2}^{\infty}\frac{F_{2n-1}^2 + 1}{F_{2n-1}^2} = \prod_{n=2}^{\infty}\frac{F_{2n+1}F_{2n-3}}{F_{2n-1}^2}.$$

Because $\lim_{m\to\infty}\frac{F_{m+j}}{F_m} = \alpha^j$, we find

$$\prod_{n=2}^{\infty}\left(1 + \frac{1}{F_{2n-1}^2}\right) = \lim_{m\to\infty}\prod_{n=2}^{m}\frac{F_{2n+1}F_{2n-3}}{F_{2n-1}^2} = \lim_{m\to\infty}\frac{F_1F_{2m+1}}{F_3F_{2m-1}} = \frac{\alpha^2}{2}.$$

For part (B), we observe that (again, using Catalan's formula)

$$\prod_{n=2}^{\infty}\left(1 - \frac{1}{F_{2n}^2}\right) = \prod_{n=2}^{\infty}\frac{F_{2n}^2 - 1}{F_{2n}^2} = \prod_{n=2}^{\infty}\frac{F_{2n+2}F_{2n-2}}{F_{2n}^2}$$

$$= \lim_{m\to\infty}\prod_{n=2}^{m}\frac{F_{2n+2}F_{2n-2}}{F_{2n}^2} = \lim_{m\to\infty}\frac{F_2F_{2m+2}}{F_4F_{2m}} = \frac{\alpha^2}{3}.$$

For part (C), we obtain

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n-1}^2}\right)\left(1+\frac{1}{F_{2n}^2}\right)=\prod_{n=2}^{\infty}\left(\frac{F_{2n-1}^2-1}{F_{2n-1}^2}\cdot\frac{F_{2n}^2+1}{F_{2n}^2}\right)$$
$$=\prod_{n=2}^{\infty}\left(\frac{F_{2n-2}F_{2n}}{F_{2n-1}^2}\cdot\frac{F_{2n-1}F_{2n+1}}{F_{2n}^2}\right)=\lim_{m\to\infty}\prod_{n=2}^{m}\frac{F_{2n-2}F_{2n+1}}{F_{2n-1}F_{2n}}=\lim_{m\to\infty}\frac{F_2F_{2m+1}}{F_3F_{2m}}=\frac{\alpha}{2}.$$

Also solved by Adilhan Ataoğlu, Brian Beasley, Brian Bradie, Charles K. Cook, Kenny B. Davenport, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Kapil Kumar Gurjar, Russell Jay Hendel, Thomas Koshy, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (student), Jason L. Smith, Albert Stadler, David Terr, Mustafa Türe (student), and the proposer.

Fun with Powers of Two

B-1265 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 58.1, February 2020)

For any integer $n \ge 1$, find a closed form expression for the sum $\displaystyle\sum_{k=1}^{n}\prod_{j=1}^{k}\left(L_{2^{j+1}}+L_{2^j}\right)$.

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

To begin, we will prove by induction that

$$\prod_{j=1}^{k}\left(L_{2^{j+1}}+L_{2^j}\right)=\frac{L_{2^{k+2}}-L_{2^{k+1}}}{4}.$$

The identity holds when $k=1$, because $L_{2^2}+L_{2^1}=7+3=\frac{47-7}{4}=\frac{L_{2^3}-L_{2^2}}{4}$. Assume it holds when $k=m$, that is, $\prod_{j=1}^{m}\left(L_{2^{j+1}}+L_{2^j}\right)=\frac{L_{2^{m+2}}-L_{2^{m+1}}}{4}$ for some integer $m\ge 1$. Then,

$$\prod_{j=1}^{m+1}\left(L_{2^{j+1}}+L_{2^j}\right)=\frac{L_{2^{m+2}}-L_{2^{m+1}}}{4}\cdot\left(L_{2^{m+2}}+F_{2^{m+1}}\right)=\frac{L_{2^{m+2}}^2-L_{2^{m+1}}^2}{4}$$
$$=\frac{\left(L_{2^{m+3}}+2\right)-\left(L_{2^{m+2}}+2\right)}{4}=\frac{L_{2^{m+3}}-L_{2^{m+2}}}{4}.$$

Thus, the identity holds for all positive integers k. Therefore,

$$\sum_{k=1}^{n}\prod_{j=1}^{k}\left(L_{2^{j+1}}+L_{2^j}\right)=\sum_{k=1}^{n}\frac{L_{2^{k+2}}-L_{2^{k+1}}}{4}=\frac{L_{2^{n+3}}-L_{2^2}}{4}=\frac{L_{2^{n+3}}-7}{4}.$$

Also solved by Brian Bradie, Robert Frontczak, G. C. Greubel, Raphael Schumacher (student), Jason L. Smith, Albert Stadler, David Terr, and the proposer.

Acknowledgment: Raphael Schumacher (student) also solved Problem B-1260, his name was inadvertently omitted by the editor.