ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2024. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1341 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Show that

$$\prod_{k=0}^{n}\left(5F_{5^k}^4 - 5F_{5^k}^2 + 1\right) = \frac{F_{5^{n+1}}}{5^{n+1}}$$

for any integer $n \geq 0$.

B-1342 Proposed by Hideyuki Othsuka, Saitama, Japan.

Given a positive integer r, and real numbers a, b such that $b \neq a\beta$, define the sequence $\{G_n\}$ by $G_0 = a$, $G_1 = b$, and $G_{n+2} = G_{n+1} + G_n$ for $n \geq 0$. If $G_{2^n r} \neq 0$ for $n \geq 1$, prove that

$$\prod_{n=1}^{\infty}\left(1 + \frac{a}{G_{2^n r}}\right) = \frac{G_{2r} + a}{F_{2r}(b - a\beta)}.$$

B-1343 Proposed by Toyesh Prakash Sharma (undergraduate), Agra College, Agra, India.

If $n \geq 3$, show that

$$\frac{1}{(F_n^2-1)(F_{n+1}^2-1)(L_n^2-1)}+\frac{1}{(F_n^2+1)(F_{n+1}^2+1)(L_n^2+1)}>\frac{2}{F_{n+1}^6}.$$

B-1344 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Let $a > b$ be positive integers such that $a^b + b^a = (2F_{3n} - 1)^{F_{3n}-1}$. Prove that $a/(F_{3n} - 1)^2$ is a positive integer, for every positive integer n.

B-1345 Proposed by Hideyuki Othsuka, Saitama, Japan.

For any integer $n \geq 0$, prove that

$$\sum_{k=0}^{n}\binom{n}{k}F_{2\lfloor k/2\rfloor}=\frac{F_{2n+1}-F_{n+2}}{2}, \qquad \text{and} \qquad \sum_{k=0}^{n}\binom{n}{k}L_{2\lfloor k/2\rfloor}=\frac{L_{2n+1}+L_{n+2}}{2}.$$

SOLUTIONS

An Infinite Sum of Reciprocals of Fibonacci Numbers

B-1321 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 61.1, February 2023)

Let $A = \{(n, r) \mid n, r \in \mathbb{N}$ with $n \geq 3$, $r \geq 1$, and, if n is even, then r is even$\}$. Prove that

$$\sum_{(n,r)\in A}\frac{1}{F_n^r}=\frac{61-15\sqrt{5}}{18}.$$

Solution by Michel Bataille, Rouen, France.

Let $S = \sum_{(n,r)\in A} \frac{1}{F_n^r}$. First, we remark that

$$S=\sum_{n=1}^{\infty}\left[\sum_{r=1}^{\infty}\left(\frac{1}{F_{2n+1}}\right)^r+\sum_{r=1}^{\infty}\left(\frac{1}{F_{2n+2}^2}\right)^r\right]=\sum_{n=1}^{\infty}\left(\frac{1}{F_{2n+1}-1}+\frac{1}{F_{2n+2}^2-1}\right).$$

Then, using $\alpha^2 - 1 = \alpha$, and $\alpha\beta = -1$, we find

$$\begin{aligned}\frac{1}{\alpha^{2n}-1}-\frac{1}{\alpha^{2n+2}-1}&=\frac{\alpha^{2n+1}}{\alpha^{4n+2}-\alpha^{2n+2}-\alpha^{2n}+1}\\&=\frac{1}{\alpha^{2n+1}-\beta^{2n+1}-(\alpha-\beta)}=\frac{1}{\sqrt{5}}\cdot\frac{1}{F_{2n+1}-1}.\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n+1}-1}=\sqrt{5}\sum_{n=1}^{\infty}\left(\frac{1}{\alpha^{2n}-1}-\frac{1}{\alpha^{2n+2}-1}\right)=\frac{\sqrt{5}}{\alpha^2-1}=\frac{5-\sqrt{5}}{2}.$$

Similar calculations yield

$$\begin{aligned}\frac{1}{\alpha^{4n}-1}-\frac{1}{\alpha^{4(n+2)}-1}&=\frac{\alpha^{4n+4}(\alpha^4-\beta^4)}{\alpha^{8n+8}-\alpha^{4n+4}(\alpha^4+\beta^4)+1}\\&=\frac{3\sqrt{5}}{\alpha^{4n+4}-7+\beta^{4n+4}}=\frac{3\sqrt{5}}{(\alpha^{2n+2}-\beta^{2n+2})^2-5}=\frac{3}{\sqrt{5}}\cdot\frac{1}{F_{2n+2}^2-1},\end{aligned}$$

so that

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n+2}^2-1}=\frac{\sqrt{5}}{3}\left(\frac{1}{\alpha^4-1}+\frac{1}{\alpha^8-1}\right).$$

Because

$$\frac{\sqrt{5}}{\alpha^4-1}=\frac{\sqrt{5}\,\beta^2}{\alpha^2-\beta^2}=\beta+1,\qquad\text{and}\qquad\frac{\sqrt{5}}{\alpha^8-1}=\frac{\sqrt{5}\,\beta^4}{\alpha^4-\beta^4}=\frac{3\beta+2}{3},$$

we have

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n+2}^2-1}=\frac{8}{9}-\frac{\sqrt{5}}{3}.$$

Thus,

$$S=\frac{5-\sqrt{5}}{2}+\frac{8}{9}-\frac{\sqrt{5}}{3}=\frac{61-15\sqrt{5}}{18}.$$

Also solved by Thomas Achammer, Brian Bradie, I. V. Fedak, Robert Frontczak, Richard Gonzalez Hernandez and Edwin Daniel Patiño Osorio (both undergraduates) (jointly), Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, Yunyong Zhang, and the proposer.

It's Easy with Jensen

B-1322 Proposed by Mihaly Bencze, Braşov, Romania, and Neculai Stanciu, Buzău, Romania.
(Vol. 61.1, February 2023)

Prove that $\dfrac{(n-1)^2}{n}\displaystyle\sum_{k=1}^{n}\left(\frac{F_k}{F_{n+2}-F_k-1}\right)^2\geq 1$ for any integer $n>1$.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

For $n>1$, let x_1, x_2, …, x_n be positive real numbers, and let

$$\mathcal{S}=\sum_{k=1}^{n}x_k.$$

Because $f(x)=\left(\frac{x}{\mathcal{S}-x}\right)^2$ is convex over the interval $(0,S)$, by Jensen's inequality,

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{x_k}{\mathcal{S}-x_k}\right)^2\geq\left(\frac{\frac{1}{n}\sum_{k=1}^{n}x_k}{\mathcal{S}-\frac{1}{n}\sum_{k=1}^{n}x_k}\right)^2=\frac{1}{(n-1)^2},$$

or

$$\frac{(n-1)^2}{n}\sum_{k=1}^{n}\left(\frac{x_k}{\mathcal{S}-x_k}\right)^2\geq 1.$$

Now, let $x_k = F_k$, so that

$$\mathcal{S} = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

Thus,

$$\frac{(n-1)^2}{n} \sum_{k=1}^{n} \left(\frac{F_k}{F_{n+2} - F_k - 1} \right)^2 \geq 1.$$

Also solved by Thomas Achammer, Michel Bataille, I. V. Fedak, Wei-Kai Lai, Edwin Daniel Patiño Osorio (undergraduate), Ángel Plaza, Albert Stadler, and the proposer.

It's Jensen Again!

B-1323 Proposed by Toyesh Prakash Sharma (undergraduate), Agra College, Agra, India.
(Vol. 61.1, February 2023)

Let n be a positive integer. Show that ${F_n}^{F_n} + {L_n}^{L_n} \geq 2{F_{n+1}}^{F_{n+1}}$.

Solution by Brian D. Beasley, Simpsonville, SC.

For $x \geq 1$, let $f(x) = x^x$. Then $f'(x) = x^x(\ln x + 1)$, so

$$f''(x) = x^{x-1} + x^x(\ln x + 1)^2.$$

Because $f''(x) > 0$ for $x \geq 1$, the function $f(x)$ is convex on $[1, \infty)$. Hence, for any λ in $(0, 1)$ and any $a \leq b$ in $[1, \infty)$, Jensen's inequality asserts that

$$\lambda f(a) + (1 - \lambda) f(b) \geq f(\lambda a + (1 - \lambda) b).$$

Taking $\lambda = 1/2$, $a = F_n$, and $b = L_n$ yields

$$F_n^{F_n} + L_n^{L_n} \geq 2 \left(\frac{F_n + L_n}{2} \right)^{\frac{F_n + L_n}{2}}.$$

Because $F_n + L_n = 2F_{n+1}$, the proof is complete.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, I. V. Fedak, Robert Frontczak, Richard Gonzalez Hernandez and Edwin Daniel Patiño Osorio (both undergraduates) (jointly), Won Kyun Jeong, Jacob Juillerat, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Henry Ricardo, Albert Stadler, David Terr, Daniel Văcaru, Andrés Ventas, and the proposer.

The Limit of the Fractional Part

B-1324 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 61.1, February 2023)

Let $\{x\}$ denote the fractional part of the real number x. Evaluate $\lim_{n\to\infty} \left\{\alpha F_{2n}^2\right\}$ and $\lim_{n\to\infty} \left\{\alpha F_{2n-1}^2\right\}$.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

For the first limit, we will first show that $\left\{\alpha F_{2n}^2\right\} = \alpha F_{2n}^2 - F_{2n}F_{2n+1} + 1$. Notice that this is equivalent to showing that

$$F_{2n}F_{2n+1} - 1 < \alpha F_{2n}^2 < F_{2n}F_{2n+1}.$$

After some algebra, we can rewrite it as

$$\alpha^{4n+1} + \beta^{4n+1} - \alpha - \beta - 5 < \alpha^{4n+1} - \beta^{4n-1} - 2\alpha < \alpha^{4n+1} + \beta^{4n+1} - \alpha - \beta,$$

which simplifies to

$$(\alpha - \beta)(1 - \beta^{4n}) - 5 < 0 < (\alpha - \beta)(1 - \beta^{4n}).$$

Because this is obviously true, our claim is established. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \left\{\alpha F_{2n}^2\right\} &= \lim_{n\to\infty} \left(\alpha F_{2n}^2 - F_{2n}F_{2n+1} + 1\right) \\
&= \lim_{n\to\infty} \frac{-\beta^{4n-1} - 2\alpha - \beta^{4n+1} + \alpha + \beta + 5}{5} \\
&= 1 - \frac{\alpha - \beta}{5} = 1 - \frac{\sqrt{5}}{5}.
\end{aligned}$$

For the second limit, we first want to show that $\left\{\alpha F_{2n-1}^2\right\} = \alpha F_{2n-1}^2 - F_{2n-1}F_{2n}$, which is equivalent to

$$F_{2n-1}F_{2n} < \alpha F_{2n-1}^2 < F_{2n-1}F_{2n} + 1.$$

After some algebra, we can rewrite it as

$$\alpha^{4n-1} + \beta^{4n-1} + \alpha + \beta < \alpha^{4n-1} - \beta^{4n-1} + 2\alpha < \alpha^{4n-1} + \beta^{4n-1} + \alpha + \beta + 5,$$

and simplify it to

$$2\beta^{4n-1} < \alpha - \beta < 2\beta^{4n-1} + 5.$$

Because this is true, the claim is established. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \left\{\alpha F_{2n}^2\right\} &= \lim_{n\to\infty} \left(\alpha F_{2n-1}^2 - F_{2n-1}F_{2n}\right) \\
&= \lim_{n\to\infty} \frac{-\beta^{4n-1} + 2\alpha - \beta^{4n+1} - \alpha - \beta}{5} \\
&= \frac{\alpha - \beta}{5} = \frac{\sqrt{5}}{5}.
\end{aligned}$$

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Brian Bradie, Dmitry Fleischman, Robert Frontczak, Richard Gonzalez Hernandez (undergraduate), Raphael Schumacher (graduate student), Albert Stadler, David Terr, Andrés Ventas, and the proposer.

Sum of Eight Consecutive Tribonacci Numbers

B-1325 Proposed by Hans J. H. Tuenter, Toronto, Canada. (Vol. 61.1, February 2023)

Let $\{\mu_n\}$ be a sequence that follows the recurrence relation $\mu_{n+3} = \mu_{n+2} + \mu_{n+1} + \mu_n$, with arbitrary initial values μ_0, μ_1, and μ_2. Prove that, for such a generalized Tribonacci sequence, the sum of eight consecutive numbers always equals four times the seventh of these numbers.

Editor's Comments: The solutions we received can be grouped into three categories. The first group used the recurrence to express the last five numbers in terms of the first three, and compute their sum explicitly for comparison against the seventh number. The second group derived the summation formula $\sum_{i=0}^{n} \mu_i = \frac{1}{2}(\mu_{n+2} + \mu_n - \mu_2 + \mu_0)$ to complete the proof. The last group, which is the largest, obtained the result directly from the recurrence relation. A few solvers managed to find a generalization.

Solution by Ralph P. Grimaldi, Rose-Hulman Institute of Technology (retired), Terre Haute, IN.

Let μ_n be a sequence that follows the recurrence relation

$$\mu_{n+k} = \mu_{n+k-1} + \mu_{n+k-2} + \cdots + \mu_n,$$

for fixed integer $k \geq 2$, with arbitrary initial values μ_0, μ_1, ..., μ_{k-1}. We claim that, for such a generalized k-bonacci sequence, the sum of $2k+2$ consecutive numbers always equals four times the $(2k+1)$st of these numbers:

$$\begin{aligned}
&\mu_{r+1} + \mu_{r+2} + \cdots + \mu_{r+2k+2} \\
&\quad = (\mu_{r+1} + \cdots + \mu_{r+k}) + \mu_{r+k+1} + (\mu_{r+k+2} + \cdots + \mu_{r+2k+1}) + \mu_{r+2k+2} \\
&\quad = 2\mu_{r+k+1} + 2(\mu_{r+k+2} + \cdots + \mu_{r+2k+1}) \\
&\quad = 2[(\mu_{r+k+1} + \mu_{r+k+2} + \cdots + \mu_{r+2k}) + \mu_{r+2k+1}] \\
&\quad = 2 \cdot 2\mu_{r+2k+1} \\
&\quad = 4\mu_{r+2k+1}.
\end{aligned}$$

Editor's Notes: Michael R. Bacon observed that the generalized result given above also holds when $k = 1$, because the sequence becomes a constant sequence, in which case the claim is trivial. Davenport reported an extension in a different direction: he found that the alternating sum of eight consecutive Tribonacci numbers is always four times the fifth of these numbers.

Also solved by Thomas Achammer, Ulkar Ahmadli (undergraduate), Michael R. Bacon, Michel Bataille, Brian D. Beasley, Brian Bradie, Charlie K. Cook, Kenny B. Davenport, Georgia Southern University Eagle Problem Solvers, Steve Edwards, I. V. Fedak, Robert Frontczak, Ian Futz (undergraduate) and Jacob Juillerat (jointly), G. C. Greubel, Won Kyun Jeong, Wei-Kai Lai and John Risher (jointly), Kathleen E. Lewis, Juha Oh (middle school student), Hideyuki Ohtsuka, Edwin Daniel Patiño Osorio (undergraduate), Ángel Plaza, Patrick Rappa, Jay L. Schiffman, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, David Terr, Eli Torak (undergraduate), Daniel Văcaru, Andrés Ventas, and the proposer.