ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2023. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1326 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let $i = \sqrt{-1}$. For any integer $n \geq 1$, prove that

$$\left| \sum_{k=1}^{n} i^k F_k \right| = F_{\lceil n/2 \rceil} \sqrt{F_{2\lfloor n/2 \rfloor + 1}}.$$

B-1327 Proposed by Brian Bradie, Christopher Newport University, Newport News, VA.

For each nonnegative integer n, define

$$a_n = \left(\sum_{k=0}^{n} F_k \right)^2 - 2\sum_{k=0}^{n} F_k^2, \qquad \text{and} \qquad b_n = \left(\sum_{k=0}^{n} L_k \right)^2 - 2\sum_{k=0}^{n} L_k^2.$$

Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{3^n}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{b_n}{3^n}$.

B-1328 **Proposed by Toyesh Prakash Sharma (undergraduate), Arga College, Arga, India.**

For any integer $n \geq 0$, show that

$$\frac{2^{n+1}F_{2n+1}}{2n+1} \geq L_n.$$

B-1329 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For any positive integer n, prove that

$$\left(\frac{F_{6n}L_{2n}}{L_{6n}F_{2n}}\right)^{L_{4n}} > e^2.$$

B-1330 **Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.**

Two sequences of numbers x_n and y_n are defined by the same building rule $z_{n+1} = \frac{1+z_n}{4-z_n}$, $n \geq 0$, but with different initial values $x_0 = 0$ and $y_0 = 1$. Prove that

$$\prod_{k=1}^{n} x_k = \frac{1}{F_{2\lfloor n/2\rfloor+2}L_{2\lfloor (n-1)/2\rfloor+3}}, \qquad \text{and} \qquad \prod_{k=1}^{n} y_k = \frac{2}{F_{2\lfloor n/2\rfloor+1}L_{2\lfloor (n-1)/2\rfloor+2}}.$$

SOLUTIONS

Yet Another Telescopic Sum

B-1306 **Proposed by Diego Rattaggi, Realgymnasium Rämibühl, Zürich, Switzerland.**
(Vol. 60.2, May 2022)

Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}F_{6n}}{(F_{2n}^2+1)^2} = 1.$$

Solution by Steve Edwards, Roswell, GA.

Using the identity $F_{a+b} + (-1)^b F_{a-b} = F_a L_b$, we obtain

$$\begin{aligned} F_{3n} &= F_{2n}L_n + (-1)^{n+1}F_n, \\ F_{2n} + (-1)^n &= F_{n-1}L_{n+1}, \\ F_{2n} + (-1)^{n+1} &= F_{n+1}L_{n-1}. \end{aligned}$$

It follows that

$$\begin{aligned} F_{3n} &= F_{2n}L_n + (-1)^{n+1}F_n \\ &= F_{2n}(F_{n-1} + F_{n+1}) + (-1)^{n+1}(F_{n+1} - F_{n-1}) \\ &= F_{n-1}[F_{2n} + (-1)^n] + F_{n+1}[F_{2n} + (-1)^{n+1}] \\ &= F_{n-1} \cdot F_{n-1}L_{n+1} + F_{n+1} \cdot F_{n+1}L_{n-1} \\ &= F_{n-1}^2 L_{n+1} + F_{n+1}^2 L_{n-1}. \end{aligned}$$

Consequently,

$$F_{6n} = F_{2n-1}^2 L_{2n+1} + F_{2n+1}^2 L_{2n-1}.$$

Next, note that Cassini's identity implies $F_{2n}^2 + 1 = F_{2n-1}F_{2n+1}$, which gives us

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}F_{6n}}{(F_{2n}^2+1)^2} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(F_{2n-1}^2 L_{2n+1} + F_{2n+1}^2 L_{2n-1})}{F_{2n-1}^2 F_{2n+1}^2} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{L_{2n+1}}{F_{2n+1}^2} + \frac{L_{2n-1}}{F_{2n-1}^2}\right).\end{aligned}$$

This is a telescoping sum with kth partial sum $L_1/F_1^2 + (-1)^{k+1}L_{2k+1}/F_{2k+1}^2$. Because $L_{2k+1}/F_{2k+1}^2 \to 0$ as $k \to \infty$, the sum equals $L_1 F_1^2$, or 1, as desired.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Kenny B. Davenport, Robert Frontczak, Hideyuki Ohtsuka, Ángel Plaza, Lucía L. Pacios and Andrés Ventas (jointly), Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, and the proposer.

It's All About Generating Function

B-1307 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.
(Vol. 60.2, May 2022)

Show that

$$\sum_{n=0}^{\infty}\left(\left\lfloor \frac{n}{2}\right\rfloor + 1\right)\frac{F_n}{2^n} = \frac{32}{5} \qquad \text{and} \qquad \sum_{n=0}^{\infty}\left(\left\lfloor \frac{n}{2}\right\rfloor + 1\right)\frac{L_n}{2^n} = 16.$$

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

For $|x| < 1$, define the (convergent) series $f(x) = \sum_{n=0}^{\infty}\left(\left\lfloor \frac{n}{2} + 1\right\rfloor\right)x^n$. Then,

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty}\left[\left(\left\lfloor \frac{2k}{2}\right\rfloor + 1\right)x^{2k} + \left(\left\lfloor \frac{2k+1}{2}\right\rfloor + 1\right)x^{2k+1}\right] \\ &= \sum_{k=0}^{\infty}\left[(k+1)x^{2k} + (k+1)x^{2k+1}\right] \\ &= (1+x)\sum_{k=0}^{\infty}(k+1)x^{2k}.\end{aligned}$$

Because

$$\sum_{k=0}^{\infty}(k+1)t^k = \frac{d}{dt}\left(\sum_{k=0}^{\infty} t^{k+1}\right) = \frac{d}{dt}\left(\frac{t}{1-t}\right) = \frac{1}{(1-t)^2},$$

we deduce that

$$f(x) = \frac{1+x}{(1-x^2)^2} = \frac{1}{(1+x)(1-x)^2}.$$

It follows that

$$f\left(\frac{\alpha}{2}\right) = \frac{1}{\left(1+\frac{\alpha}{2}\right)\left(1-\frac{\alpha}{2}\right)^2} = \frac{8}{(2+\alpha)(2-\alpha)^2}.$$

From $\alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$, $\alpha\beta = -1$, $\alpha + \beta = 1$, and $\alpha - \beta = \sqrt{5}$, we gather that

$$2 + \alpha = \alpha^2 + 1 = \alpha^2 - \alpha\beta = \alpha(\alpha - \beta) = \alpha\sqrt{5},$$

and

$$2 - \alpha = 1 + \beta = \beta^2.$$

We find

$$f\left(\frac{\alpha}{2}\right) = \frac{8}{\sqrt{5}\,\alpha\beta^4} = \frac{8\alpha^3}{\sqrt{5}}.$$

In a similar manner, we also find $f\left(\frac{\beta}{2}\right) = -\frac{8\beta^3}{\sqrt{5}}$. We obtain

$$\sum_{n=0}^{\infty}\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)\frac{F_n}{2^n} = \frac{1}{\sqrt{5}}\left[f\left(\frac{\alpha}{2}\right) - f\left(\frac{\beta}{2}\right)\right] = \frac{8(\alpha^3 + \beta^3)}{5} = \frac{8L_3}{5} = \frac{32}{5},$$

and

$$\sum_{n=0}^{\infty}\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)\frac{L_n}{2^n} = f\left(\frac{\alpha}{2}\right) + f\left(\frac{\beta}{2}\right) = \frac{8(\alpha^3 - \beta^3)}{\sqrt{5}} = 8F_3 = 16.$$

Editor's Notes: Greubel further generalized the results to

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)\frac{F_{n+m}}{a^n} &= \frac{a^3\left[(a^2+a-2)F_{m+1} + (a^3-a^2-2a+3)F_m\right]}{(a^4-3a^2+1)(a^2-a-1)}\\
\sum_{n=0}^{\infty}\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)\frac{L_{n+m}}{a^n} &= \frac{a^3\left[(a^2+a-2)L_{m+1} + (a^3-a^2-2a+3)L_m\right]}{(a^4-3a^2+1)(a^2-a-1)}\\
\sum_{n=0}^{\infty}\left(\left\lfloor\frac{n}{2}\right\rfloor + n\right)\frac{F_{n+m}}{a^n} &= \frac{2a\left[(a^3-a+1)F_{m+2} + (a-1)(a+2)F_{m+1}\right]}{(a^4-3a^2+1)(a^2-a-1)}\\
\sum_{n=0}^{\infty}\left(\left\lfloor\frac{n}{2}\right\rfloor + n\right)\frac{L_{n+m}}{a^n} &= \frac{2a\left[(a^3-a+1)L_{m+2} + (a-1)(a+2)L_{m+1}\right]}{(a^4-3a^2+1)(a^2-a-1)}.
\end{aligned}$$

The desired results follow by letting $m = 0$ and $a = 2$.

Also solved by Ulrich Abel, Thomas Achammer, Michel Bataille, Brian Bradie, Charles K. Cook, Nandan Sai Dasireddy, Kenny B. Davenport, Steve Edwards, Eagle Problem Solvers, Dmitry Fleischman, G. C. Greubel, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, Seán M. Stewart, David Terr, Andrés Ventas, and the proposer.

Yet Another Telescopic Sum, Again!

B-1308 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**
(Vol. 60.2, May 2022)

Evaluate

$$\sum_{n=0}^{\infty}\frac{1}{L_{2F_n}L_{2F_{n+1}}L_{2F_{n+2}}L_{2F_{n+3}}}\left(\frac{1}{L_{2F_n}} - \frac{1}{L_{2F_{n+3}}}\right).$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

With the product formula $L_sL_t = L_{s+t} + (-1)^s L_{t-s}$, we obtain

$$L_{2F_{n+1}}L_{2F_{n+2}} = L_{2F_{n+3}} + L_{2F_n}.$$

It follows that

$$\begin{aligned}
\frac{1}{L_{2F_n}L_{2F_{n+1}}L_{2F_{n+2}}L_{2F_{n+3}}}\left(\frac{1}{L_{2F_n}} - \frac{1}{L_{2F_{n+3}}}\right) &= \frac{(L_{2F_{n+3}} - L_{2F_n})L_{2F_{n+1}}L_{2F_{n+2}}}{L^2_{2F_n}L^2_{2F_{n+1}}L^2_{2F_{n+2}}L^2_{2F_{n+3}}} \\
&= \frac{(L_{2F_{n+3}} - L_{2F_n})(L_{2F_{n+3}} + L_{2F_n})}{L^2_{2F_n}L^2_{2F_{n+1}}L^2_{2F_{n+2}}L^2_{2F_{n+3}}} \\
&= \frac{L^2_{2F_{n+3}} - L^2_{2F_n}}{L^2_{2F_n}L^2_{2F_{n+1}}L^2_{2F_{n+2}}L^2_{2F_{n+3}}} \\
&= \frac{1}{L^2_{2F_n}L^2_{2F_{n+1}}L^2_{2F_{n+2}}} - \frac{1}{L^2_{2F_{n+1}}L^2_{2F_{n+2}}L^2_{2F_{n+3}}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{L_{2F_n}L_{2F_{n+1}}L_{2F_{n+2}}L_{2F_{n+3}}}\left(\frac{1}{L_{2F_n}} - \frac{1}{L_{2F_{n+3}}}\right) \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{1}{L^2_{2F_n}L^2_{2F_{n+1}}L^2_{2F_{n+2}}} - \frac{1}{L^2_{2F_{n+1}}L^2_{2F_{n+2}}L^2_{2F_{n+3}}}\right) \\
&\quad= \frac{1}{L^2_{2F_0}L^2_{2F_1}L^2_{2F_2}} - \lim_{n\to\infty}\frac{1}{L^2_{2F_{n+1}}L^2_{2F_{n+2}}L^2_{2F_{n+3}}} \\
&\quad= \frac{1}{L_0^2L_2^4} = \frac{1}{324}.
\end{aligned}$$

Also solved by Thomas Achammer, Michel Bataille, Kenny B. Davenport, Won Kyun Jeong, Ángel Plaza, and the proposer.

A New Result From an Old Problem

B-1309 Proposed by Kenny B. Davenport, Dallas, PA. (Vol. 60.2, May 2022)

Prove that, for any integer $n \geq 1$,

$$\sum_{k=1}^{n} F_kF^2_{k+1} = \frac{3F^3_n + 3F^3_{n+1} + F^3_{n+2} - F_{3n+2} - 3}{6}.$$

Solution by Michel Bataille, Rouen, France.

We use the result proved in Problem B-1235 (Vol. 57, No. 3, p. 282):

$$\sum_{k=1}^{n} F_{k-1}F_kF_{k+1} = \frac{1}{3}\left(F^3_{n-1} + F^3_n + F^3_{n+1} - \frac{F_{3n-1}+3}{2}\right) \tag{1}$$

and the following identity:

$$F^3_{n+1} + F^3_n - F^3_{n-1} = F_{3n}. \tag{2}$$

Proof of (2): Because $\alpha\beta = -1$, we have

$$F_n^3 = \frac{(\alpha^n - \beta^n)^3}{5\sqrt{5}} = \frac{\alpha^{3n} - 3(-1)^n\alpha^n + 3(-1)^n\beta^n - \beta^{3n}}{5\sqrt{5}} = \frac{F_{3n} - 3(-1)^n F_n}{5},$$

from which we deduce that

$$\begin{aligned} F_{n+1}^3 + F_n^3 - F_{n-1}^3 &= \frac{1}{5}\left(F_{3n+3} + F_{3n} - F_{3n-3}\right) \\ &= \frac{1}{5}\left[(3F_{3n} + 2F_{3n-1}) + F_{3n} - (2F_{3n-1} - F_{3n})\right] \\ &= F_{3n}. \end{aligned}$$

Now, let $S_n = \sum_{k=1}^n F_k F_{k+1}^2$ and $T_n = \sum_{k=1}^n F_k^2 F_{k+1}$. From (1), we obtain

$$S_n - T_n = \frac{1}{3}\left(F_{n-1}^3 + F_n^3 + F_{n+1}^3 - \frac{F_{3n-1}+3}{2}\right)$$

and

$$S_n + T_n = \sum_{k=1}^{n} F_k F_{k+1} F_{k+2} = \sum_{k=1}^{n+1} F_{k-1} F_k F_{k+1} = \frac{1}{3}\left(F_n^3 + F_{n+1}^3 + F_{n+2}^3 - \frac{F_{3n+2}+3}{2}\right).$$

Therefore,

$$\begin{aligned} S_n &= \frac{1}{6}\left(F_{n-1}^3 + 2F_n^3 + 2F_{n+1}^3 + F_{n+2}^3 - 3 - \frac{F_{3n-1} + F_{3n+2}}{2}\right) \\ &= \frac{1}{6}\left((3F_n^3 + 3F_{n+1}^3 + F_{n+2}^3) - (F_{n+1}^3 + F_n^3 - F_{n-1}^3) - 3 - F_{3n+2} + \frac{F_{3n+2} - F_{3n-1}}{2}\right) \\ &= \frac{1}{6}\left(3F_n^3 + 3F_{n+1}^3 + F_{n+2}^3 - F_{3n+2} - 3\right), \end{aligned}$$

where the last equality follows from (2) and $\frac{F_{3n+2}-F_{3n-1}}{2} = F_{3n}$.

Also solved by Thomas Achammer, Brian Beasley, Brian Bradie, Steve Edwards, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Won Kyun Jeong, Wei-Kai Lai and John Risher (graduate student) (jointly), Hideyuki Ohtsuka, Ángel Plaza, Jason L. Smith, Albert Stadler, Andrés Ventas, and the proposer.

A Relative of Another Old Problem

B-1310 Proposed by Steve Edwards, Roswell, GA. (Vol. 60.2, May 2022)

For any positive integer n, find a closed form expression for the sum

$$\sum_{k=1}^{n} \left\lfloor \frac{F_k}{\alpha F_k - F_{k-1}} \right\rfloor.$$

Solution by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

This problem is a complement to Problem B-1260, which asked the readers to find a closed form expression for the sum $\sum_{k=1}^{n}\left\lfloor\frac{F_k}{\alpha F_k - F_{k+1}}\right\rfloor$. We start with $\alpha^k = \alpha F_k + F_{k-1}$. This gives

$$\alpha F_k - F_{k-1} = \alpha^k - 2F_{k-1} = \alpha^k - L_k + F_k = F_k - \beta^k.$$

Now, if k is odd, then $\beta^k < 0$; thus,

$$0 < F_k < F_k - \beta^k.$$

If k is even, then $\beta^k > 0$; because $2\beta^k < F_k$ (which holds or all k), we have

$$F_k - \beta^k < F_k < 2(F_k - \beta^k).$$

Hence,

$$\left\lfloor\frac{F_k}{\alpha F_k - F_{k-1}}\right\rfloor = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ 1, & \text{if } k \text{ is even.} \end{cases}$$

Consequently,

$$\sum_{k=1}^{n}\left\lfloor\frac{F_k}{\alpha F_k - F_{k-1}}\right\rfloor = \left\lfloor\frac{n}{2}\right\rfloor.$$

Using the same arguments, we can generalize the result to all odd m according to

$$\sum_{k=1}^{n}\left\lfloor\frac{F_{mk}}{\alpha F_{mk} - F_{mk-1}}\right\rfloor = \left\lfloor\frac{n}{2}\right\rfloor.$$

Editor's Notes: Plaza obtained an analog for the Lucas numbers:

$$\sum_{k=1}^{n}\left\lfloor\frac{L_{k+1}}{\alpha L_{k+1} - L_k}\right\rfloor = \left\lfloor\frac{n}{2}\right\rfloor.$$

Also solved by Thomas Achammer, Michel Bataille, Brian Beasley, Brian Bradie, Charles K. Cook and Michael R. Bacon (jointly), Kenny B. Davenport, Eagle Problem Solvers, Dmitry Fleischman, G. C. Greubel, Hideyuki Ohtsuka, Ángel Plaza, Albert Stadler, David Terr, Andrés Ventas, and the proposer.