ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2024. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1336 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Show the following identities:

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{L_{4n}+18} = \frac{1}{8}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{F_{4n}}{(L_{4n}+18)^2} = \frac{9}{800}.$$

B-1337 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that

(i) $\dfrac{F_n^3}{F_{n-1}} + \dfrac{F_{n+2}^3}{F_n} - \dfrac{F_{n+1}^4}{F_{n-1}F_n} = 2F_{n+1}F_{n+2}$, for any integer $n \geq 3$,

(ii) $\dfrac{L_n^3}{L_{n-1}} + \dfrac{L_{n+2}^3}{L_n} - \dfrac{L_{n+1}^4}{L_{n-1}L_n} = 2L_{n+1}L_{n+2}$, for any integer $n \geq 1$.

<u>B-1338</u> Proposed by Quang Hung Tran, Hanoi National University of Education High School for Gifted Students, Hanoi, Vietnam.

Prove that, for any integer $n \geq 0$,

(a) $\dfrac{1}{F_{n+1}} + \dfrac{1}{F_{n+2}} > \dfrac{16}{9L_{n+1} - 16F_n}$,

(b) $\dfrac{1}{L_{n+1}} + \dfrac{1}{L_{n+2}} > \dfrac{16}{45F_{n+1} - 16L_n}$.

<u>B-1339</u> Proposed by Michel Bataille, Rouen, France.

Let n be a positive integer. Prove that

$$\left\lfloor \sqrt{2\left(F_{n+1}^2 - F_{n+1}F_{n-1} + F_{n-1}^2\right)} \right\rfloor - \left\lfloor \sqrt{F_{n+1}F_{n-1}} \right\rfloor = F_n.$$

<u>B-1340</u> Proposed by Hans J. H. Tuenter, Toronto, Canada.

Let a, b, and n be integers, with n nonnegative, and x any real or complex number. Evaluate

$$\sum_{i=0}^{n} \binom{n}{i} F_{a-1}^{n-i}(x) F_a^i(x) F_{b+i}(x),$$

where $F_n(x)$ are the Fibonacci polynomials, defined by the recurrence relation $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. Note that the Fibonacci polynomials are defined at negative indices by extending the above recurrence relation and that they satisfy the relation $F_{-n}(x) = (-1)^{n+1}F_n(x)$.

SOLUTIONS

<u>Special Cases of Jensen's Inequality</u>

<u>B-1313</u> (Corrected) Proposed by Daniel Văcaru, Economical College Maria Teiuleanu, Piteşti, Romania, and Mihály Bencze, Aprily Lajos, Braşov, Romania.
(Vol. 60.4, November 2022)

For $a \leq -1$, show that

$$\sum_{k=1}^{n} F_k (F_{n+2} - F_k - 1)^a \geq \left(\frac{n-1}{n}\right)^a (F_{n+2} - 1)^{a+1},$$

$$\sum_{k=1}^{n} F_k^2 (F_n F_{n+1} - F_k^2)^a \geq \left(\frac{n-1}{n}\right)^a (F_n F_{n+1})^{a+1}.$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Because $\sum_{k=1}^{n} F_k = F_{n+2} - 1$, and $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, both inequalities are particular cases of the following more general inequality.

Let x_k, for $k = 1, 2, \ldots, n$, be positive real numbers, and define $S_n = \sum_{k=1}^{n} x_k$. For $a \le -1$, we claim that

$$\sum_{k=1}^{n} x_k (S_n - x_k)^a \ge \left(\frac{n-1}{n}\right)^a S_n^{a+1}.$$

Because we can rewrite it as

$$\sum_{k=1}^{n} \frac{x_k}{S_n}\left(1 - \frac{x_k}{S_n}\right)^a \ge \left(\frac{n-1}{n}\right)^a,$$

we may assume that $S_n = 1$. The function $f(x) = x(1-x)^a$ is convex for $x \in (0,1)$ because $f''(x) = a(1-x)^{a-2}(-2 + x + ax) > 0$. Then, by Jensen's inequality,

$$\sum_{k=1}^{n} x_k (1 - x_k)^a = \sum_{k=1}^{n} f(x_k) \ge n f\left(\frac{1}{n}\right) = n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^a = \left(\frac{n-1}{n}\right)^a,$$

which completes the proof.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Dmitry Fleischman, G. C. Greubel, Wei-Kai Lai, Albert Stadler, Andrés Ventas, and the proposers.

An Infinite Sum of Arctangent

B-1316 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 60.4, November 2022)

Prove that

$$\sum_{n=0}^{\infty} \tanh^{-1} \frac{1}{L_{2\cdot 3^n}} = \frac{1}{4} \ln 5.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA.

Note that

$$\tanh^{-1} \frac{1}{\alpha^{2\cdot 3^n}} - \tanh^{-1} \frac{1}{\alpha^{2\cdot 3^{n+1}}} = \tanh^{-1}\left(\frac{\frac{1}{\alpha^{2\cdot 3^n}} - \frac{1}{\alpha^{6\cdot 3^n}}}{1 - \frac{1}{\alpha^{8\cdot 3^n}}}\right)$$

$$= \tanh^{-1}\left(\frac{\alpha^{2\cdot 3^n} - \frac{1}{\alpha^{2\cdot 3^n}}}{\alpha^{4\cdot 3^n} - \frac{1}{\alpha^{4\cdot 3^n}}}\right) = \tanh^{-1} \frac{F_{2\cdot 3^n}}{F_{4\cdot 3^n}} = \tanh^{-1} \frac{1}{L_{2\cdot 3^n}},$$

so the given series telescopes. In particular,

$$\sum_{n=0}^{\infty} \tanh^{-1} \frac{1}{L_{2\cdot 3^n}} = \sum_{n=0}^{\infty}\left(\tanh^{-1} \frac{1}{\alpha^{2\cdot 3^n}} - \tanh^{-1} \frac{1}{\alpha^{2\cdot 3^{n+1}}}\right) = \tanh^{-1} \frac{1}{\alpha^2}$$

$$= \frac{1}{2} \ln \frac{1 + \frac{1}{\alpha^2}}{1 - \frac{1}{\alpha^2}} = \frac{1}{2} \ln \frac{\alpha^2 - \alpha\beta}{\alpha^2 + \alpha\beta} = \frac{1}{2} \ln \frac{\alpha - \beta}{\alpha + \beta} = \frac{1}{2} \ln \sqrt{5} = \frac{1}{4} \ln 5.$$

Solution 2 by Kristen Hartz (undergraduate), PennWest University, California, PA.

We use $\tanh^{-1} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $|x|<1$ to obtain the following:

$$\sum_{n=0}^{\infty}\tanh^{-1}\frac{1}{L_{2\cdot 3^n}} = \lim_{m\to\infty}\sum_{n=0}^{m}\frac{1}{2}\ln\left(\frac{1+\frac{1}{L_{2\cdot 3^n}}}{1-\frac{1}{L_{2\cdot 3^n}}}\right) = \frac{1}{2}\lim_{m\to\infty}\sum_{n=0}^{m}\ln\left(\frac{L_{2\cdot 3^n}+1}{L_{2\cdot 3^n}-1}\right)$$

$$= \frac{1}{2}\lim_{m\to\infty}\sum_{n=0}^{m}\left[\ln(L_{2\cdot 3^n}+1)-\ln(L_{2\cdot 3^n}-1)\right]$$

$$= \frac{1}{2}\lim_{m\to\infty}\left[\ln\left(\prod_{k=0}^{m}\left(L_{2\cdot 3^k}+1\right)\right)-\ln\left(\prod_{k=0}^{m}\left(L_{2\cdot 3^k}-1\right)\right)\right].$$

Using the identities $\prod_0^{n-1}\left(L_{2\cdot 3^k}-1\right)=F_{3^n}$ and $\prod_0^{n-1}\left(L_{2\cdot 3^k}+1\right)=L_{3^n}$ (Identities 126 and 127 [1, pg. 93], respectively), we obtain

$$\sum_{n=0}^{\infty}\tanh^{-1}\frac{1}{L_{2\cdot 3^n}} = \frac{1}{2}\lim_{m\to\infty}\ln\frac{L_{3^{m+1}}}{F_{3^{m+1}}} = \frac{1}{2}\ln\sqrt{5} = \frac{1}{4}\ln 5.$$

Reference

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

Also solved by Thomas Achammer, Michel Bataille, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Won Kyun Jeong, Ángel Plaza, Jason L. Smith (two solutions), Albert Stadler, David Terr, Dan Weiner, and the proposer.

A Depressed Quartic Equation with Fibonacci Coefficients

B-1317 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine. (Vol. 60.4, November 2022)

For every integer n, find the real roots of the equation

$$x^4 - 2F_{n+1}^2x^2 - 4F_{n+1}F_{n-1}^2x + F_{n+1}^4 - F_{n-1}^4 = 0.$$

Solution by Hans J. H. Tuenter, Toronto, Canada.

Consider the depressed quartic polynomial $p(x) = x^4 - 2b^2x^2 - 4ba^2x + b^4 - a^4$, where a and b are arbitrary constants. This can be written as

$$p(x) = [(x-b)^2 - a^2]\,[(x+b)^2 + a^2].$$

Hence, its roots are $b \pm a$ and $-b \pm ai$. For $b = F_{n+1}$ and $a = F_{n-1}$, one finds the real roots as $F_{n+1} - F_{n-1} = F_n$ and $F_{n+1} + F_{n-1} = L_n$, when $n \neq 1$. When $n = 1$, we have $a = 0$ and obtain the real roots ± 1, each with multiplicity two.

For the Lucas equivalent

$$x^4 - 2L_{n+1}^2x^2 - 4L_{n+1}L_{n-1}^2x + L_{n+1}^4 - L_{n-1}^4 = 0,$$

one finds the real roots as L_n and $5F_n$.

Editor's Note: It is clear from the solution that the complex roots are $-F_{n+1} \pm F_{n-1}i$, for the given equation (when $n \neq 1$), and $-L_{n+1} \pm L_{n-1}i$, for the Lucas equivalent.

Also solved by Thomas Achammmer, Emily Antrim (undergraduate), Michael R. Bacon and Charles K. Cook (jointly), Michel Bataille, Brian D. Beasley, Brian Bradie, Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, Alison Gallegos (undergraduate), G. C. Greubel, Ralph P. Grimaldi, Kristen Hartz (undergraduate), Peyton Matheson (high school student), Hideyuki Ohtsuka, Ángel Plaza, Jason L. Smith, Albert Stadler, David Terr, Andrés Ventas, and the proposer.

An Intriguing Alternating Sum

B-1318 Proposed by Kenny B. Davenport, Dallas, PA. (Vol. 60.4, November 2022)

Prove that

$$\sum_{k=1}^{n}(-1)^k F_{k-1}F_kF_{k+1} = \frac{(-1)^n}{12}\left(-4F_n^3 - F_{n+1}^3 + 2F_{n+2}^3 + 4F_{n+3}^3 - F_{n+4}^3\right) - \frac{1}{2}.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan.

We shall use the following identities:

(i) $F_{n+4}^3 - 3F_{n+3}^2 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 = 0$ (see [1], page 108);

(ii) $3F_{n+1}F_nF_{n-1} = F_{n+1}^3 - F_n^3 - F_{n-1}^3$ (see [1], page 111).

Let

$$a_n = (-1)^n\left(-4F_n^3 - F_{n+1}^3 + 2F_{n+2}^3 + 4F_{n+3}^3 - F_{n+4}^3\right).$$

We find, by (i) and (ii)

$$\begin{aligned} a_n - a_{n-1} &= (-1)^n\left(-4F_{n-1}^3 - 5F_n^3 + F_{n+1}^3 + 6F_{n+2}^3 + 3F_{n+3}^2 - F_{n+4}^3\right) \\ &= (-1)^n\left(-4F_{n-1}^3 - 5F_n^3 + F_{n+1}^3 + 6F_{n+2}^3 + 3F_{n+3}^3 - F_{n+4}^3\right. \\ &\qquad \left. + F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3\right) \\ &= 4\,(-1)^n\left(F_{n+1}^3 - F_n^3 - F_{n-1}^3\right) \\ &= 12\,(-1)^n\,F_{n+1}F_nF_{n-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{k=1}^{n}(-1)^k F_{k-1}F_kF_{k+1} &= \frac{1}{12}\sum_{k=1}^{n}(a_k - a_{k-1}) = \frac{1}{12}\,(a_n - a_0) \\ &= \frac{1}{12}\,(-1)^n\left(-4F_n^3 - F_{n+1}^3 + 2F_{n+2}^3 + 4F_{n+3}^3 - F_{n+4}^3\right) - \frac{1}{2}. \end{aligned}$$

Editor's Notes: Ohtsuka also reported that he found a simple formula

$$\sum_{k=1}^{n}(-1)^k F_{k-1}F_kF_{k+1} = \frac{(-1)^n F_{3n+1} + 4F_{n+2} - 5}{10}.$$

Tuenter found a generalization. His results, along with some historical notes, appear in an article in this issue of *The Fibonacci Quarterly*.

Reference

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Volume 1, John Wiley & Sons, New York, 2017.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Charles K. Cook and Michael R. Bacon (jointly), Steve Edwards, G. C. Greubel, Won Kyun Jeong, Wei-Kai Lai, Yinhao Liu (undergraduate), Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, Hans J. H. Tuenter, and the proposer.

An Inequality with Fibonacci and Lucas Radicals

B-1319 Proposed by Toyesh Prakash Sharma (student), St. C. F. Andrews School, Agra, India.
(Vol. 60.4, November 2022)

Show that, for $n \geq 4$,

$$F_n^{\frac{1}{F_n}} L_n^{\frac{1}{L_n}} \geq \left(F_{n+1}^2\right)^{\frac{1}{F_{n+1}}}.$$

Solution by Michel Bataille, Rouen, France.

The second derivative of the function $f(x) = \frac{\ln x}{x}$ satisfies $f''(x) = \frac{2\ln x - 3}{x^3}$, hence is nonnegative for $x \geq e^{3/2}$. It follows that f is convex on the interval $\left[e^{3/2}, \infty\right)$. Now, if $n \geq 5$, then $L_n \geq F_n \geq 5 \geq e^{3/2}$, hence

$$\frac{\ln(F_n)}{F_n} + \frac{\ln(L_n)}{L_n} = f(F_n) + f(L_n)$$
$$\geq 2\, f\left(\frac{F_n + L_n}{2}\right) = 2\left(\frac{\ln\left(\frac{F_n+L_n}{2}\right)}{\frac{F_n+L_n}{2}}\right) = \frac{2}{F_{n+1}} \cdot \ln(F_{n+1}),$$

because $F_n + L_n = 2F_{n+1}$. By exponentiation, we obtain the desired inequality

$$F_n^{\frac{1}{F_n}} L_n^{\frac{1}{L_n}} \geq (F_{n+1}^2)^{\frac{1}{F_{n+1}}}.$$

If $n = 4$, the inequality also holds because $\sqrt[3]{3} \cdot \sqrt[7]{7} > \sqrt[5]{25}$ (as it is readily checked). We conclude that the inequality holds for $n \geq 4$.

Also solved by Thomas Achammer, Brian Bradie, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Ángel Plaza, Wei-Kai Lai, Albert Stadler, David Terr, Andrés Ventas, Nicuşor Zlota, and the proposer.

The Difference of Two Infinite Fibonacci Series

B-1320 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 60.4, November 2022)

Prove that

$$\sum_{n=1}^{\infty} \frac{4}{(F_n F_{n+3})^2} = \sum_{n=1}^{\infty} \frac{1}{(F_n F_{n+2})^2} + \frac{1}{4}.$$

Solution by Won Kyun Jeong, Kyungpook National University, Daegu, South Korea.

It follows from the identity

$$F_{n+3}^2 - F_n^2 = (F_{n+3} + F_n)(F_{n+3} - F_n) = 4F_{n+1}F_{n+2}$$

that

$$\frac{4}{(F_nF_{n+3})^2} = \frac{1}{F_{n+1}F_{n+2}}\left(\frac{1}{F_n^2} - \frac{1}{F_{n+3}^2}\right) = \frac{F_{n+2}}{F_n^2F_{n+1}F_{n+2}^2} - \frac{F_{n+1}}{F_{n+1}^2F_{n+2}F_{n+3}^2}.$$

Hence, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{4}{(F_nF_{n+3})^2} &= \sum_{n=1}^{\infty}\left(\frac{F_{n+2}}{F_n^2F_{n+1}F_{n+2}^2} - \frac{F_{n+1}}{F_{n+1}^2F_{n+2}F_{n+3}^2}\right)\\
&= \frac{F_3}{F_1^2F_2F_3^2} + \sum_{n=1}^{\infty}\left(-\frac{F_{n+1}}{F_{n+1}^2F_{n+2}F_{n+3}^2} + \frac{F_{n+3}}{F_{n+1}^2F_{n+2}F_{n+3}^2}\right)\\
&= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{F_{n+1}^2F_{n+3}^2}\\
&= \frac{1}{2} + \left(\sum_{n=1}^{\infty}\frac{1}{F_n^2F_{n+2}^2}\right) - \frac{1}{F_1^2F_3^2}\\
&= \sum_{n=1}^{\infty}\frac{1}{(F_nF_{n+2})^2} + \frac{1}{4}.
\end{aligned}$$

This completes the proof.

Also solved by Thomas Achammer, Michel Bataille, Brian Bradie, Robert Frontczak, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, and the proposer.