ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at flucamatmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-665 Proposed by G. C. Greubel, Newport News, VA

Given the bilateral series

$$ {}_1H_1(a;b;x) = \sum_{n=-\infty}^{n=\infty} \frac{(a)_n}{(b)_n} x^n $$

derive general expressions that reduce to the equations

$$ \sum_{r=0}^{1}\sum_{n=0}^{\infty} \binom{2m}{2n} {}_1H_1(2n-2m; 2n+1; (-1)^r\sqrt{5}) = 5\cdot 4^{m-1} L_{2m} $$

$$ \sum_{r=0}^{1}\sum_{n=0}^{\infty} (-1)^r \binom{2m}{2n} {}_1H_1(2n-2m; 2n+1; (-1)^{r+1}\sqrt{5}) = 5^{3/2}\cdot 4^{m-1} F_{2m}. $$

H-666 Proposed by H.-J. Seiffert, Berlin, Germany

The Pell and Pell-Lucas numbers are defined by $P_0 = 0,\ P_1 = 1,\ Q_0 = 2,\ Q_1 = 2$ and $P_{n+1} = 2P_n + P_{n-1},\ Q_{n+1} = 2Q_n + Q_{n-1}$ for all $n \geq 1$. Prove that, for all positive integers n, we have

$$ P_{2n-1} = 2^{n-2}(4^{n-1}+1) - 2^{2-n} \sum_{k=0}^{\lfloor (n-3)/4 \rfloor} \binom{4n-2}{2n-8k-5}, $$

$$ Q_{2n} = 2^{n}(2^{2n-1}+1) - 2^{3-n} \sum_{k=0}^{\lfloor (n-2)/4 \rfloor} \binom{4n}{2n-8k-4}. $$

H-667 Proposed by Herman Roelants, Leuven, Belgium

Let $u_n = pu_{n-1} + qu_{n-2}$ for all $n \geq 2$, with $u_0 = 0,\ u_1 = 1$ and $p,\ q > 0$. Prove that

$$ \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n u_{2n+1}}{(2n+1)(p^2+4q)^n} t^{2n+1} \qquad \text{with} \qquad t = \frac{2}{1+\sqrt{\frac{p^2+8q}{p^2+4q}}}. $$

H-668 Proposed by A. Cusumano, Great Neck, NY

For each $k \geq 2$, *let* $(F_n^{(k)})_{n \geq 1}$ *be the* k*th order linear recurrence given by*

$$F_{n+k}^{(k)} = \sum_{i=0}^{k-1} F_{n+i}^{(k)}, \quad for\ all\ n \geq 1,$$

with $F_n^{(k)} = 1$ *for* $n = 1, \ldots, k$. *Prove the following:*
(a) $R_k = \lim_{n \to \infty} F_{n+1}^{(k)}/F_n^{(k)}$ *exists for all* $k \geq 1$.
(b) $\lim_{k \to \infty} R_k = 2$.
(c) $\lim_{k \to \infty}(R_{k+1} - R_k)/(R_{k+2} - R_{k+1}) = 2$.

SOLUTIONS

An Identity For The Lucas Numbers

H-648 Proposed by Ovidiu Furdui, Kalamazoo, MI
(Vol. 44, No. 4, November 2006)

Let n be a positive integer. Prove that the following identity holds

$$L_{n+1} = (n+1)\left(\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{\lfloor \frac{n-1}{2} \rfloor - j + 1} \cdot \binom{\lfloor \frac{n}{2} \rfloor + j}{\lfloor \frac{n-1}{2} \rfloor - j}\right) + 1.$$

Solution by H.-J. Seiffert, Berlin, Germany

The Lucas polynomials are defined by $L_0(x) = 2$, $L_1(x) = x$, and $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$ for $n \geq 0$. We shall prove that, for all complex numbers x and all nonnegative integers n,

$$L_{n+1}(x) = x^{n+1} + \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{n+1}{\lfloor \frac{n-1}{2} \rfloor - j + 1} \binom{\lfloor \frac{n}{2} \rfloor + j}{\lfloor \frac{n-1}{2} \rfloor - j} x^{2j+(1+(-1)^n)/2}.$$

The proposed identity is obtained when letting $x = 1$.

It is known (see equation (2.16) of [1]) that

$$L_{n+1}(x) = x^{n+1} + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{n+1}{n+1-k} \binom{n+1-k}{k} x^{n+1-2k}.$$

Since

$$\frac{1}{n+1-k}\binom{n+1-k}{k} = \frac{1}{k}\binom{n-k}{k-1}, \qquad k = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor,$$

$$\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \qquad n - 1 - \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor, \quad \text{and}$$

$$n - 1 - 2\left\lfloor \frac{n-1}{2} \right\rfloor = (1 + (-1)^n)/2,$$

it is easily verified that the above stated identity follows by reindexing $k = \lfloor \frac{n-1}{2} \rfloor - j + 1$.

[1] A. F. Horadam and Bro. J. M. Mahon, *Pell and Pell-Lucas Polynomials*, The Fibonacci Quarterly, 23.1 (1985), 7–20.

Also solved by the proposer.

Fibonacci Secants

H-649 Proposed by Stanley Rabinowitz, Chelmsford, MA (Vol. 44, No. 4, November 2006)

Find positive integers a, b, and c such that $\sec(F_a) + \sec(F_b) = F_c$, where all angles are measured in degrees.

Solution by the proposer

We start with the known numerical value

$$\cos 72^\circ = \frac{\sqrt{5}-1}{4}.$$

Then using $\cos 2x = 2\cos^2 x - 1$, we get

$$\cos 144^\circ = 2\left(\frac{\sqrt{5}-1}{4}\right)^2 - 1 = -\frac{\sqrt{5}+1}{4}.$$

Thus,

$$\sec 72^\circ = \frac{1}{\cos 72^\circ} = \frac{4}{\sqrt{5}-1} = \sqrt{5}+1,$$

and

$$\sec 144^\circ = \frac{1}{\cos 144^\circ} = -\frac{4}{\sqrt{5}+1} = 1-\sqrt{5}.$$

Hence,

$$\sec 72^\circ + \sec 144^\circ = 2 = F_3.$$

We recognize 144 as the Fibonacci number F_{12}, but at first glance, 72 does not appear to be a Fibonacci number. However, modulo 360, 72 is a Fibonacci number. In particular,

$$F_{96} = 51680708854858323072 = 143557524596828675 \cdot 360 + 72.$$

Thus,

$$\sec F_{12} + \sec F_{96} = F_3,$$

so $a = 12,\ b = 96,\ c = 3$ is a solution to our problem.

Secants, Cosecants and Differentials

H-650 Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 45, No. 1, February 2007)

Let $D = d/dz$ be the differential operator. Let $f = f(z) = \csc z$, where z is any complex number $\neq n\pi$, where n is any integer. Prove the following identities valid for all integers $m \geq 1$:

$$f^{2m+2} = \frac{1}{(2m+1)!}\prod_{n=1}^{m}\{D^2 + 4n^2\}(f^2);$$

$$f^{2m+1} = \frac{1}{(2m)!}\prod_{n=1}^{m}\{D^2 + (2n-1)^2\}(f).$$

Show that the same relations hold with the function $f(z) = \sec z$ when $z \neq (n + 1/2)\pi$, where n is any integer.

Solution by Michael S. Becker and Charles K. Cook, Sumter, SC

First note that the restrictions on the complex numbers is for the $\csc z$ and $\sec z$ to be defined. Note also that for $f(z) = \csc z$ or $f(z) = \sec z$ the following routinely derived formulas are valid:

$$D^2 f^2 = 6f^4 - 4f^2, \qquad D^2 f = 2f^3 - f,$$
$$D^2 f^{2m+2} = (2m+3)(2m+2)f^{2m+4} - (2m+2)^2 f^{2m+2}, \qquad \text{and}$$
$$D^2 f^{2m+1} = (2m+2)(2m+1)f^{2m+3} - (2m+1)^2 f^{2m+1}.$$

In what follows, we use the above formulas as needed. If $m = 1$, then

$$\frac{1}{(2m+1)!}\prod_{n=1}^{m}\{D^2 + 4n^2\}f^2 = \frac{1}{6}\{D^2 + 4\}f^2 = \frac{1}{6}\{6f^4 - 4f^2 + 4f^2\} = f^4.$$

If for fixed $m \ge 1$,

$$\frac{1}{(2m+1)!}\prod_{n=1}^{m}\{D^2 + 4n^2\}f^2 = f^{2m+2},$$

then for $m+1$ we have

$$\frac{1}{(2m+3)!}\prod_{n=1}^{m+1}\{D^2 + 4n^2\}f^2 = \frac{1}{(2m+3)(2m+2)}\{D^2 + 4(m+1)^2\}f^{2m+2}$$
$$= \frac{1}{(2m+3)(2m+2)}\{(2m+3)(2m+2)f^{2m+4} - (2m+2)^2 f^{2m+2} + 4(m+1)^2 f^{2m+2}\}$$
$$= f^{2m+4} = f^{2(m+1)+2}.$$

Thus, by induction, in either case $f(z) = \csc z$ or $f(z) = \sec z$, we have

$$f^{2m+2} = \frac{1}{(2m+1)!}\prod_{n=1}^{m}\{D^2 + 4n^2\}f^2.$$

Similarly, if $m = 1$, then

$$\frac{1}{(2m)!}\prod_{n=1}^{m}\{D^2 + (2n-1)^2\}f^2 = \frac{1}{2}\{D^2 + 1\}f = \frac{1}{2}\{2f^3 - f + f\} = f^3.$$

If for fixed $m \ge 1$,

$$\frac{1}{(2m)!}\prod_{n=1}^{m}\{D^2 + (2n-1)^2\}f^2 = f^{2m+1},$$

then for $m+1$ we have

$$\frac{1}{(2m+2)!}\prod_{n=1}^{m+1}\{D^2 + (2n-1)^2\}f^2 = \frac{1}{(2m+2)(2m+1)}\{D^2 + (2m+1)^2\}f^{2m+1}$$
$$= \frac{1}{(2m+2)(2m+1)}\{(2m+2)(2m+1)f^{2m+3} - (2m+1)^2 f^{2m+1} + (2m+1)^2 f^{2m+1}\}$$
$$= f^{2m+3} = f^{2(m+1)+1}.$$

Thus, again by induction, in either case $f(z) = \csc z$ or $f(z) = \sec z$, we have

$$f^{2m+1} = \frac{1}{(2m)!}\prod_{n=1}^{m}\{D^2 + (2n-1)^2\}f.$$

Also solved by Kenneth Davenport, G. C. Greubel, and the proposer.

Late Acknowledgement. H-647 was also solved by Paul S. Bruckman and by Rigoberto Florez and Charles K. Cook (jointly).

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