ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-868 Proposed by Juan Lopez Gonzalez, Madrid, Spain

Prove that if N is an odd perfect number, then it satisfies

$$\frac{\sigma_0(N)\ln 2}{2} = N\ln 2 - \sum_{\substack{d|N\\ d>1}} \sum_{k=1}^{(d-1)/2} \sum_{\ell\geq 1} \frac{k^{2\ell}(2^{2\ell}-1)}{\ell 2^{2\ell}}\zeta(2\ell),$$

where $\sigma_0(N)$ is the number of divisors of N and for $k > 1$, $\zeta(k)$ is the Riemann zeta function.

H-869 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For positive integer n, prove that

$$\sum_{k=1}^{n}(-1)^k L_k F_k^5 = \frac{(-1)^n(F_n^5 F_{n+3} - F_n^2)}{2}.$$

H-870 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For any positive integer n, find closed form expressions for the sums

$$\text{(i)} \quad \sum_{k=1}^{n}(L_{F_k}L_{F_{k+1}})(F_{F_k}F_{F_{k+1}})^3 \qquad \text{and} \qquad \text{(ii)} \quad \sum_{k=1}^{n}(F_{F_k}F_{F_{k+1}})(L_{F_k}L_{F_{k+1}})^3.$$

H-871 Proposed by Robert Frontczak, Stuttgart, Germany

Let $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ be the balancing and Lucas-balancing numbers, respectively, i.e., $B_{n+1} = 6B_n - B_{n-1}$ and $C_{n+1} = 6C_n - C_{n-1}$ for all $n \geq 1$ and $B_0 = 0$, $B_1 = 1$, $C_0 = 1$, $C_1 = 3$. Show that

$$\sum_{n=1}^{\infty} \frac{B_n}{n(n+1)6^n} = 6\ln 6 - \frac{17}{\sqrt{8}}\ln(3+\sqrt{8}) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{C_n}{n(n+1)6^n} = 1 - 17\ln 6 + 6\sqrt{8}\ln(3+\sqrt{8}).$$

H-872 Proposed by Robert Frontczak, Stuttgart, Germany

Prove that

$$\sum_{n=1}^{\infty} \eta(2n)\frac{F_{2n}}{5^n} = \frac{\pi}{10\cos(\frac{\pi}{2\sqrt{5}})} \quad \text{and} \quad \sum_{n=1}^{\infty} \eta(2n)\frac{L_{2n}}{5^n} = \frac{\pi}{2\cos(\frac{\pi}{2\sqrt{5}})} - 1,$$

where $\eta(s) = \sum_{k=1}^{\infty}(-1)^{k-1}/k^s$ (defined for $\mathrm{Re}(s) > 0$) is the Dirichlet η (or alternating Riemann zeta) function.

SOLUTIONS

Closed formulas for some sums of products of balancing numbers

H-834 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 57, No. 1, February 2018)

Let $\{B_n\}_{n\in\mathbb{Z}}$ and $\{C_n\}_{n\in\mathbb{Z}}$ denote the balancing and Lucas-balancing numbers, respectively, given by

$$B_{n+1} = 6B_n - B_{n-1} \quad \text{and} \quad C_{n+1} = 6C_n - C_{n-1} \quad \text{for all} \quad n \geq 1,$$

with $B_0 = 0$, $B_1 = 1$, $C_0 = 1$, $C_1 = 3$. Prove that for integers $n \geq 1$, $j \geq 0$

(i) $\sum_{k=1}^{n} C_{k\mp j}B_{k\pm j} = \frac{1}{32}(C_{2n+1} - 3) \pm \frac{n}{2}B_{2j}$;
(ii) $\sum_{k=1}^{n} C_{k-j}C_{k+j}B_{k-j}B_{k+j} = \frac{1}{768}(B_{4n+2} - 6(2n+1)) - \frac{n}{4}B_{2j}^2$.

Solution by Ángel Plaza, Gran Canaria, Spain

We will use Binet's formulas for these numbers, $B_n = \dfrac{\alpha^n - \beta^n}{4\sqrt{2}}$ and $C_n = \dfrac{\alpha^n + \beta^n}{2}$, where $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$. Note that $\alpha\beta = 1$.

Therefore, for (i)

$$\begin{aligned}
\sum_{k=1}^{n} C_{k\mp j}B_{k\pm j} &= \frac{1}{8\sqrt{2}}\sum_{k=1}^{n}\left(\alpha^{k\mp j} + \beta^{k\mp j}\right)\left(\alpha^{k\pm j} - \beta^{k\pm j}\right) \\
&= \frac{1}{8\sqrt{2}}\sum_{k=1}^{n}\left(\alpha^{2k} - \beta^{2k}\right) + \frac{1}{8\sqrt{2}}\sum_{k=1}^{n}\left(\left(\frac{\alpha}{\beta}\right)^{\pm j} - \left(\frac{\beta}{\alpha}\right)^{\pm j}\right) \\
&= \frac{1}{8\sqrt{2}}\left(\frac{\alpha^2 - \alpha^{2n+2}}{1-\alpha^2} - \frac{\beta^2 - \beta^{2n+2}}{1-\beta^2}\right) + \frac{n}{2}\left(\frac{\left(\frac{\alpha}{\beta}\right)^{\pm j} - \left(\frac{\beta}{\alpha}\right)^{\pm j}}{4\sqrt{2}}\right) \\
&= \frac{1}{8\sqrt{2}}\left(\frac{\alpha - \alpha^{2n+1}}{\beta - \alpha} - \frac{\beta - \beta^{2n+1}}{\alpha - \beta}\right) \pm \frac{n}{2}B_{2j} \\
&= \frac{-\alpha - \beta + \alpha^{2n+1} + \beta^{2n+1}}{8\sqrt{2}\cdot 4\sqrt{2}} \pm \frac{n}{2}B_{2j} \\
&= \frac{1}{32}(C_{2n+1} - 3) \pm \frac{n}{2}B_{2j}.
\end{aligned}$$

Now, for (ii), we use that for any integer m, $C_m B_m = \dfrac{C_{2m}}{4\sqrt{2}}$, so

$$\begin{aligned}
\sum_{k=1}^{n} C_{k-j}C_{k+j}B_{k-j}B_{k+j} &= \frac{1}{(4\sqrt{2})^2}\sum_{k=1}^{n} C_{2k-2j}C_{2k+2j}\\
&= \frac{1}{32\cdot 4}\sum_{k=1}^{n}\left(\alpha^{2k-2j}+\beta^{2k-2j}\right)\left(\alpha^{2k+2j}+\beta^{2k+2j}\right)\\
&= \frac{1}{128}\sum_{k=1}^{n}\left(\alpha^{4k}+\beta^{4k}+\left(\frac{\alpha}{\beta}\right)^{2j}+\left(\frac{\beta}{\alpha}\right)^{2j}\right)\\
&= \frac{1}{128}\left(\frac{\alpha^4-\alpha^{4n+4}}{1-\alpha^4}+\frac{\beta^4-\beta^{4n+4}}{1-\beta^4}+n\left(\alpha^{4j}+\beta^{4j}\right)\right)\\
&= \frac{1}{128}\left(\frac{\alpha^2-\alpha^{4n+2}}{\beta^2-\alpha^2}+\frac{\beta^2-\beta^{4n+2}}{\alpha^2-\beta^2}+n\left(32B_{2j}^2+2\right)\right)\\
&= \frac{1}{768}\left(-\frac{\alpha^2+\beta^2+\alpha^{4n+2}-\beta^{4n+2}}{\alpha-\beta}+6n\left(32B_{2j}^2+2\right)\right)\\
&= \frac{1}{768}(B_{4n+2}-6(2n+1))-\frac{n}{4}B_{2j}^2.
\end{aligned}$$

Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, Hideyuki Ohtsuka, and the proposer.

Identities between higher order Bernoulli numbers and Stirling numbers

H-835 Proposed by Andrei K. Svinin and Svetlana V. Svinina, Matrosov Institute for System Dynamics and Control Theory of SB RAS, Irkutsk, Russia (Vol. 57, No. 1, February 2019)

Let $B_q^{(k)}$ be the higher order Bernoulli numbers that are defined by an exponential generating function as

$$\frac{t^k}{(e^t-1)^k}=\sum_{q\ge 0}\frac{B_q^{(k)}}{q!}t^q.$$

Prove that

$$B_n^{(k)}=\sum_{q=1}^{n}\frac{s(q+k,k)}{\binom{q+k}{k}}S(n,q),$$

where $s(n,k)$ and $S(n,k)$ are the Stirling numbers of the first and second type, respectively.

Solution by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany

We apply the exponential generating functions

$$\log^m(1+x)=m!\sum_{n=m}^{\infty}s(n,m)\frac{x^n}{n!}\quad\text{and}\quad(e^x-1)^m=m!\sum_{n=m}^{\infty}S(n,m)\frac{x^n}{n!}$$

of the Stirling numbers of the first and second kind, respectively. Putting $t = \log(1+x)$, such that $|\log(1+x)| < 2\pi$, we obtain

$$\begin{aligned} \frac{t^k}{(e^t-1)^k} &= \frac{\log^k(1+x)}{x^k} = k! \sum_{j=0}^{\infty} s(j+k,k) \frac{x^j}{(j+k)!} \\ &= \sum_{j=0}^{\infty} s(j+k,k) \binom{j+k}{k}^{-1} \frac{(e^t-1)^j}{j!} \\ &= \sum_{j=0}^{\infty} s(j+k,k) \binom{j+k}{k}^{-1} \sum_{n=j}^{\infty} S(n,j) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{n} \binom{j+k}{k}^{-1} s(j+k,k) S(n,j). \end{aligned}$$

This proves that

$$\frac{t^k}{(e^t-1)^k} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}$$

with coefficients

$$B_n^{(k)} = \sum_{j=0}^{n} \binom{j+k}{k}^{-1} s(j+k,k) S(n,j).$$

Remark 1. Note that $S(n,0) = 0$ for $n \in \mathbb{N}$. If the sum starts with $j = 0$, it is correct also in the case $n = 0$.

Remark 2. In the special case $k = 1$, we obtain a representation of the Bernoulli numbers

$$B_n = B_n^{(1)} = \sum_{j=0}^{n} (-1)^j \frac{j!}{j+1} S(n,j)$$

in terms of Stirling numbers of the second kind. Here, we used $s(j+1,1) = (-1)^j j!$ for $j \in \mathbb{N} \cup \{0\}$.

Remark 3. In [1], we find formula (2.2):

$$(x-1)(x-2)\cdots(x-m) = \sum_{n=0}^{m} \binom{m}{n} B_n^{(m+1)} x^{m-n},$$

which is cited from Chapter 6 of the book [2]. Comparison with

$$\begin{aligned} x(x-1)(x-2)\cdots(x-m) &= \sum_{j=0}^{m+1} s(m+1,j) x^j, \\ (x-1)(x-2)\cdots(x-m) &= \sum_{j=0}^{m+1} s(m+1,m+1-j) x^{m-j} \end{aligned}$$

yields, for $k \in \mathbb{N}$, the initial Bernoulli numbers of higher order

$$B_n^{(k)} = \binom{k-1}{n}^{-1} s(k,k-n) \qquad (n = 0, \ldots, k-1).$$

Another view is, for fixed n,

$$\begin{aligned} B_0^{(k)} &= 1 \qquad (k \geq 0), \\ B_1^{(k)} &= \frac{1}{k-1} s(k, k-1) = -k/2 \qquad (k \geq 1). \end{aligned}$$

More of such formulas can be found on Page 146 of [2].

[1] L. Carlitz, *Some theorems on Bernoulli numbers of higher order*, Pacific J. Math., **2** (1952), 127–139.
[2] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, 1924.

Also solved by Khristo N. Boyadzhiev, Dmitry Fleischman, Won Kyun Jeong, and the proposers.

Closed formulas for sums of products of members from a certain sequence

H-836 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 57, No. 1, February 2019)

Given a real number $p > 0$, define the sequence $\{S_n\}_{n\geq 0}$ by

$$S_0 = p, \quad S_n = S_{n-1}^2 + p \quad \text{for} \quad n \geq 1.$$

For any integer $n \geq 0$, find closed form expressions for the sums

$$\text{(i)} \quad \sum_{k=0}^{n} S_k S_{k+1} \cdots S_n \qquad \text{and} \qquad \text{(ii)} \quad \sum_{k=0}^{n} (S_k S_{k+1} \cdots S_n)^2.$$

Solution by Raphael Schumacher, ETH Zurich, Switzerland

We will prove by induction that

$$\sum_{k=0}^{n} S_k S_{k+1} \cdots S_n = \frac{S_{n+1}}{p} - 1 = \frac{S_n^2}{p} \quad \forall n \in \mathbb{N}_0,$$

and that

$$\sum_{k=0}^{n} (S_k S_{k+1} \cdots S_n)^2 = \frac{S_n^4 + 2pS_n^2}{p(p+2)} = \frac{S_{n+1}^2 - p^2}{p(p+2)} \quad \forall n \in \mathbb{N}_0.$$

The above two formulas are true for $n = 0$, because we have

$$p = S_0 = \sum_{k=0}^{0} S_k S_{k+1} \cdots S_n = \frac{S_{0+1}}{p} - 1 = \frac{S_1}{p} - 1 = \frac{p^2 + p}{p} - 1 = p = \frac{p^2}{p} = \frac{S_0^2}{p}$$

and

$$\begin{aligned} p^2 &= S_0^2 = \sum_{k=0}^{0} (S_k S_{k+1} \cdots S_n)^2 = \frac{S_0^4 + 2pS_0^2}{p(p+2)} = \frac{p^4 + 2p^3}{p(p+2)} \\ &= \frac{(p^2+p)^2 - p^2}{p(p+2)} = \frac{S_{0+1}^2 - p^2}{p(p+2)} = \frac{S_1^2 - p^2}{p(p+2)}. \end{aligned}$$

We assume that the first formula

$$\sum_{k=0}^{n} S_k S_{k+1} \cdots S_n = \frac{S_{n+1}}{p} - 1$$

is correct for $n \in \mathbb{N}_0$ and show that this implies the correctness of the formula for $n+1 \in \mathbb{N}$.

We have

$$\begin{aligned} \sum_{k=0}^{n+1} S_k S_{k+1} \cdots S_n &= \left(\sum_{k=0}^{n} S_k S_{k+1} \cdots S_n\right) S_{n+1} + S_{n+1} = \left(\frac{S_{n+1}}{p} - 1\right) S_{n+1} + S_{n+1} \\ &= \frac{S_{n+1}^2}{p} - S_{n+1} + S_{n+1} = \frac{S_{n+1}^2}{p} = \frac{S_{n+2} - p}{p} = \frac{S_{n+2}}{p} - 1 = \frac{S_{(n+1)+1}}{p} - 1 \end{aligned}$$

for all $n \in \mathbb{N}_0$. The formula

$$\sum_{k=0}^{n} S_k S_{k+1} \cdots S_n = \frac{S_{n+1}}{p} - 1 = \frac{S_{n+1} - p}{p} = \frac{(S_n^2 + p) - p}{p} = \frac{S_n^2}{p} \quad \forall n \in \mathbb{N}_0$$

is equivalent and also true. If the second formula

$$\sum_{k=0}^{n} (S_k S_{k+1} \cdots S_n)^2 = \frac{S_n^4 + 2pS_n^2}{p(p+2)}$$

is correct for $n \in \mathbb{N}_0$, then this implies that the formula is also correct for $n+1 \in \mathbb{N}$, because

$$\begin{aligned} \sum_{k=0}^{n+1} (S_k S_{k+1} \cdots S_n)^2 &= \left(\sum_{k=0}^{n} (S_k S_{k+1} \cdots S_n)^2\right) S_{n+1}^2 + S_{n+1}^2 = \left(\frac{S_n^4 + 2pS_n^2}{p(p+2)}\right) S_{n+1}^2 + S_{n+1}^2 \\ &= \left(\frac{S_n^4 + 2pS_n^2}{p(p+2)} + 1\right) S_{n+1}^2 = \left(\frac{S_n^4 + 2pS_n^2 + p^2 + 2p}{p(p+2)}\right) S_{n+1}^2 \\ &= \left(\frac{(S_n^2 + p)^2 + 2p}{p(p+2)}\right) S_{n+1}^2 = \left(\frac{S_{n+1}^2 + 2p}{p(p+2)}\right) S_{n+1}^2 = \frac{S_{n+1}^4 + 2pS_{n+1}^2}{p(p+2)} \end{aligned}$$

for all $n \in \mathbb{N}_0$ and it holds also that

$$\sum_{k=0}^{n} (S_k S_{k+1} \cdots S_n)^2 = \frac{S_n^4 + 2pS_n^2}{p(p+2)} = \frac{(S_n^2 + p)^2 - p^2}{p(p+2)} = \frac{S_{n+1}^2 - p^2}{p(p+2)} \quad \forall n \in \mathbb{N}_0.$$

Also solved by Dmitry Fleischman and the proposer.

Relations among sums of Tribonacci numbers

H-837 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 57, No. 2, May 2019)

The Tribonacci numbers $\{T_n\}_{n\geq 0}$ satisfy $T_0 = 0$, $T_1 = T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \geq 3$. Prove that for any $n \geq 1$

$$\sum_{k=1}^{n} T_{2(n-k)+2} \left(\sum_{j=0}^{2(n-k)} T_j\right) = \frac{1}{2}\left(\left(\sum_{k=1}^{n} T_{2k}\right)^2 - \left(\sum_{k=1}^{n} T_{2k-1}\right)^2\right).$$

Solution by Hideyuki Ohtsuka, Saitama, Japan

The given identity can be rewritten as follows

$$\sum_{k=0}^{n-1} T_{2k+2} \sum_{j=0}^{2k} T_j = \frac{1}{2}\left(\sum_{k=1}^{2n} T_k\right)\left(\sum_{k=1}^{2n} (-1)^k T_k\right).$$

Here, using the identities

$$\sum_{k=1}^{n} T_k = \frac{T_{n+2}+T_n-1}{2} \quad \text{and} \quad \sum_{k=1}^{n} (-1)^k T_k = \frac{(-1)^n(T_{n+1}-T_{n-1})-1}{2}$$

(see [1]), we have

$$\sum_{k=0}^{n-1} T_{2k+2}(T_{2k+2}+T_{2k}-1) = \frac{1}{4}\left(T_{2n+2}+T_{2n}-1\right)\left(T_{2n+1}-T_{2n-1}-1\right). \tag{1}$$

The proof of (1) is by induction on n. For $n=1$, we have the left side and right side of (1) equal 0. We assume that (1) holds for n. For $n+1$, we have

$$\begin{aligned} &\sum_{k=0}^{n} T_{2k+2}(T_{2k+2}+T_{2k}-1) = T_{2n+2}(T_{2n+2}+T_{2n}-1) + \sum_{k=0}^{n-1} T_{2k+2}(T_{2k+2}+T_{2k}-1) \\ = \; & T_{2n+2}(T_{2n+2}+T_{2n}-1) + \frac{1}{4}(T_{2n+2}+T_{2n}-1)(T_{2n+1}-T_{2n-1}-1) \\ = \; & \frac{1}{4}(4T_{2n+2}+T_{2n+1}-T_{2n-1}-1)(T_{2n+2}+T_{2n}-1) \\ = \; & \frac{1}{4}(T_{2n+4}+T_{2n+2}-1)(T_{2n+3}-T_{2n+1}-1) \qquad \text{because} \end{aligned}$$

$$\begin{aligned} & (4T_{2n+2}+T_{2n+1}-T_{2n-1}-1)-(T_{2n+4}+T_{2n+2}-1) = 3T_{2n+2}+T_{2n+1}-T_{2n-1}-T_{2n+4} \\ = \; & 3T_{2n+2}+T_{2n+1}-T_{2n-1}-(T_{2n+3}+T_{2n+2}+T_{2n+1}) = -T_{2n+3}+2T_{2n+2}-T_{2n-1} \\ = \; & -(T_{2n+2}+T_{2n+1}+T_{2n})+2T_{2n+2}-T_{2n-1} = T_{2n+2}-T_{2n+1}-T_{2n}-T_{2n-1} = 0, \end{aligned}$$

and

$$(T_{2n+2}+T_{2n}-1)-(T_{2n+3}-T_{2n+1}-1) = T_{2n+2}+T_{2n+1}+T_{2n}-T_{2n+3} = 0.$$

Thus, (1) holds for $n+1$. Therefore, (1) is proved.

[1] R. Frontczak, *Sums of Tribonacci and Tribonacci-Lucas numbers*, Internat. J. Math. Analysis, **12** (2018), 19–24.

Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, Raphael Schumacher, and the proposer.

Sums with Lucas numbers and binomial coefficients

H-838 Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain (Vol. 57, No. 2, May 2019)

Find a closed form expression for the following sum, where $r > 1$ and $n \geq r$ are integers

$$\sum_{j=0}^{n-r}\left(\binom{r+j}{r}-\binom{r+j-1}{r}-\binom{r+j-2}{r}\right)L_{n-(r+j)}.$$

Solution by Brian Bradie, Newport News, VA

We find a closed form expression for the more general sum

$$\sum_{j=0}^{n-r}\left(\binom{r+j}{j}-\binom{r+j-1}{j}-\binom{r+j-2}{j}\right)G_{n-(r+j)},$$

where $\{G_n\}_{n\geq 0}$ is the generalized Fibonacci sequence with $G_0 = a$, $G_1 = b$, and $G_n = G_{n-1}+G_{n-2}$ for $n \geq 2$. Now,

$$\begin{aligned}&\sum_{j=0}^{n-r}\left(\binom{r+j}{j}-\binom{r+j-1}{j}-\binom{r+j-2}{j}\right)G_{n-(r+j)}\\ =&\sum_{j=0}^{n-r-2}\binom{r+j}{j}(G_{n-r-j}-G_{n-r-j-1}-G_{n-r-j-2})+\binom{n-1}{r}(G_1-G_0)+\binom{n}{r}G_0\\ =&\binom{n-1}{r}(b-a)+\binom{n}{r}a=\frac{(n-1)!}{r!(n-r)!}(ar+b(n-r)).\end{aligned}$$

Now for the Lucas sequence $a = 2$ and $b = 1$, we have

$$\sum_{j=0}^{n-r}\left(\binom{r+j}{r}-\binom{r+j-1}{r}-\binom{r+j-2}{r}\right)L_{n-(r+j)}=\frac{(n-1)!}{r!(n-r)!}(n+r).$$

Also solved by Dmitry Fleischman, Hideyuki Ohtsuka, and the proposers.

Late acknowledgement: Dmitry Fleischman has solved Advanced Problem **H-833**.

Errata: In Advanced Problem **H-854** the correct limit to compute is

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{\frac{L_n}{(x+1)L_{n+1}}}-(f(x))^{\frac{L_n}{xL_{n+1}}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right).$$