ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-782 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Given positive integers r and s find formulas for the sums

$$\text{(i)} \sum_{n=1}^{\infty} \frac{(-1)^{srn}}{\alpha^{(s-1)rn} F_{rn} F_{r(n+1)} F_{r(n+2)} \cdots F_{r(n+s)}};$$

$$\text{(ii)} \sum_{n=1}^{\infty} \frac{(-1)^{srn}}{\alpha^{(s-1)rn} L_{rn} L_{r(n+1)} L_{r(n+2)} \cdots L_{r(n+s)}}.$$

H-783 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\text{(i)} \sum_{n=1}^{\infty} \frac{1}{F_n^2+1} = \frac{-3+5\sqrt{5}}{6};$$

$$\text{(ii)} \sum_{n=3}^{\infty} \frac{1}{F_n^2-1} = \frac{43-15\sqrt{5}}{18};$$

$$\text{(iii)} \sum_{n=3}^{\infty} \frac{1}{F_n^4-1} = \frac{35-15\sqrt{3}}{18}.$$

H-784 Proposed by Gleb Glebov, Simon Fraser University, Canada.

Prove that

$$\text{(i)} \sum_{k=1}^{\infty} \left[\frac{1}{24k+11} - \frac{1}{24k-11} + \frac{1}{24k+1} - \frac{1}{24k-1}\right] = \frac{\pi(\sqrt{6}+\sqrt{2})}{12} - \frac{12}{11};$$

$$\text{(ii)} \sum_{k=1}^{\infty} \left[\frac{1}{24k+7} - \frac{1}{24k-7} + \frac{1}{24k+5} - \frac{1}{24k-5}\right] = \frac{\pi(\sqrt{6}-\sqrt{2})}{12} - \frac{12}{35}.$$

H-785 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $m \geq n \geq 1$, find closed forms expressions for the sums

(i) $\displaystyle\sum_{k=0}^{n} F_{2k} \binom{2n}{n+k}_F \binom{2m}{m+k}_F$;

(ii) $\displaystyle\sum_{k=0}^{n} F_{2k} \binom{2n}{n+k}_F^{-1} \binom{2m}{m+k}_F^{-1}$.

H-786 **Proposed by Atara Shriki, Oranim College of Education.**

Assume that the consecutive numbers in the Fibonacci sequence are the coordinates of a polygon's vertices in the Cartesian coordinate system, counterclockwise:

$$A_1(F_1, F_2);\ A_2(F_3, F_4);\ A_3(F_5, F_6);\ A_4(F_7, F_8);\ \ldots;\ A_n(F_{2n-1}, F_{2n}).$$

What is the area of such a polygon?

SOLUTIONS

Sums of Products of Fibonacci Numbers and Binomial Coefficients

H-752 **Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.**
(Vol. 52, No. 2, May 2014)

Prove that

$$(1) 5^m L_{2m+1} \sum_{p=0}^{2n+1} \binom{2n+1}{p} \sum_{k=0}^{p} \binom{p}{k} F_k = 5^n L_{2n+1} \sum_{p=0}^{2m+1} \binom{2m+1}{p} \sum_{k=0}^{p} \binom{p}{k} F_k,$$

$$(2) 5^m F_{2m+1} \sum_{p=0}^{2n+1} \binom{2n+1}{p} \sum_{k=0}^{p} \binom{p}{k} L_k = 5^n F_{2n+1} \sum_{p=0}^{2m+1} \binom{2m+1}{p} \sum_{k=0}^{p} \binom{p}{k} L_k.$$

Solution by Ángel Plaza, Gran Canaria, Spain.

Both identities straightforwardly come from the fact that the binomial transform of the Fibonacci sequence is the bisection of the Fibonacci sequence, that is $\sum_{k=0}^{p} \binom{p}{k} F_k = F_{2p}$ and the binomial transform of the Lucas sequence is the bisection of the Lucas sequence, that is $\sum_{k=0}^{p} \binom{p}{k} L_k = L_{2p}$. We will show only identity (1). We use LHS and RHS, respectively for the left-hand side and right-hand side of (1). Then,

$$\begin{aligned} LHS &= 5^m L_{2m+1} \sum_{p=0}^{2n+1} \binom{2n+1}{p} F_{2p} = 5^m L_{2m+1} \sum_{p=0}^{2n+1} \binom{2n+1}{p} \frac{\alpha^{2p} - \beta^{2p}}{\sqrt{5}} \\ &= 5^m L_{2m+1} \frac{\left(1+\alpha^2\right)^{2n+1} - \left(1+\beta^2\right)^{2n+1}}{\sqrt{5}} \\ &= 5^m L_{2m+1} \frac{(2+\alpha)^{2n+1} - (2+\beta)^{2n+1}}{\sqrt{5}} \\ &= 5^m L_{2m+1} 5^n L_{2n+1} = 5^{n+m} L_{2m+1} 5^n L_{2n+1}, \end{aligned}$$

since $2+\alpha=\frac{5+\sqrt{5}}{2}=\sqrt{5}\alpha$ and $2+\beta=\frac{5-\sqrt{5}}{2}=-\sqrt{5}\beta$.

Similarly $RHS=5^{n+m}L_{2n+1}L_{2m+1}$ and hence (1) holds.

Also solved by Kenneth B. Davenport, Zbigniew Jakubczyk, Harris Kwong, Hideyuki Ohtsuka, and the proposers.

Sums of Fourth Powers of Fibonacci Numbers with Indices in Arithmetic Progressions

H-753 Proposed by H. Ohtsuka, Saitama, Japan. (Vol. 52, No. 2, May 2014)

For integers $n\geq 1,\ m\geq 1,\ a\neq 0$ and b, prove that

$$\sum_{k=1}^{n}F_{ak+b}^{4m}=\sum_{r=1}^{2m}\binom{4m}{2m-r}\frac{(-1)^{(an+b+1)r}F_{anr}L_{(an+a+2b)r}}{25^m F_{ar}}+\binom{4m}{2m}\frac{n}{25^m}.$$

Solution by Harris Kwong, SUNY, Fredonia.

We deduce from

$$\begin{aligned}(\alpha^{ak+b}-\beta^{ak+b})^{4m}&=\sum_{i=0}^{4m}\binom{4m}{i}(-1)^i\alpha^{(ak+b)(4m-i)}\beta^{(ak+b)i}\\&=\binom{4m}{2m}+\sum_{r=1}^{2m}\binom{4m}{2m-r}(-1)^{2m-r}\alpha^{(ak+b)(2m+r)}\beta^{(ak+b)(2m-r)}\\&\quad+\sum_{r=1}^{2m}\binom{4m}{2m+r}(-1)^{2m+r}\alpha^{(ak+b)(2m-r)}\beta^{(ak+b)(2m+r)}\\&=\binom{4m}{2m}+\sum_{r=1}^{2m}\binom{4m}{2m-r}(-1)^r(\alpha\beta)^{2(ak+b)m}\left(\frac{\alpha}{\beta}\right)^{(ak+b)r}\\&\quad+\sum_{r=1}^{2m}\binom{4m}{2m+r}(-1)^r(\alpha\beta)^{2(ak+b)m}\left(\frac{\beta}{\alpha}\right)^{(ak+b)r}\\&=\binom{4m}{2m}+\sum_{r=1}^{2m}\binom{4m}{2m-r}(-1)^r\left[\left(\frac{\alpha}{\beta}\right)^{ak+b)r}+\left(\frac{\beta}{\alpha}\right)^{(ak+b)r}\right]\end{aligned}$$

that

$$25^m\sum_{k=1}^{n}F_{ak+b}^{4m}=\binom{4m}{2m}n+\sum_{r=1}^{2m}\binom{4m}{2m-r}(-1)^r\sum_{k=1}^{n}\left[\left(\frac{\alpha}{\beta}\right)^{(ak+b)r}+\left(\frac{\beta}{\alpha}\right)^{(ak+b)r}\right].$$

We find

$$\sum_{k=1}^{n}\left(\frac{\alpha}{\beta}\right)^{(ak+b)r} = \left(\frac{\alpha}{\beta}\right)^{(a+b)r}\frac{1-\left(\frac{\alpha}{\beta}\right)^{arn}}{1-\left(\frac{\alpha}{\beta}\right)^{ar}}$$
$$= \left(\frac{\alpha}{\beta}\right)^{(a+b)r}\cdot\frac{\beta^{ar}}{\beta^{arn}}\cdot\frac{\beta^{arn}-\alpha^{arn}}{\beta^{ar}-\alpha^{ar}}$$
$$= \frac{\alpha^{(a+b)r}}{\beta^{(an+b)r}}\cdot\frac{F_{arn}}{F_{ar}}.$$

In a similar manner, we also find

$$\sum_{k=1}^{n}\left(\frac{\beta}{\alpha}\right)^{(ak+b)r} = \frac{\beta^{(a+b)r}}{\alpha^{(an+b)r}}\cdot\frac{F_{arn}}{F_{ar}}.$$

Therefore,

$$\sum_{k=1}^{n}\left[\left(\frac{\alpha}{\beta}\right)^{ak+b)r}+\left(\frac{\beta}{\alpha}\right)^{(ak+b)r}\right] = \frac{F_{arn}}{F_{ar}}\left(\frac{\alpha^{(a+b)r}}{\beta^{(an+b)r}}+\frac{\beta^{(a+b)r}}{\alpha^{(an+b)r}}\right)$$
$$= \frac{F_{arn}}{F_{ar}}\cdot\frac{\alpha^{(a+2b+an)r}+\beta^{(a+2b+an)r}}{(\alpha\beta)^{(an+b)r}}$$
$$= \frac{(-1)^{(an+b)r}F_{arn}L_{(a+2b+an)r}}{F_{ar}},$$

from which the desired result follows immediately.

Also solved by the proposer.

Identities with Tribonacci like Sequences

H-754 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 52, No. 1, February 2014)

Let a, b and n be integers. The two sequences $\{T_n\}$ and $\{S_n\}$ satisfy

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{with arbitrary } T_0,\ T_1,\ T_2,$$
$$S_{n+3} = S_{n+2} + S_{n+1} + S_n \quad \text{with } S_0 = 3,\ S_1 = 1,\ S_2 = 3$$

for all integers n. Let $R_n = S_n + 1$. For $n \ge 1$, prove that

$$(R_a^2 - R_{-a}^2)\sum_{k=1}^{n} T_{ak+b}^2 = A_n - A_0,$$

where

$$A_n = 2T_{an+b}(R_aT_{an+a+b} + R_{-a}T_{an-a+b}) - (T_{an+a+b} - T_{an-a+b})^2 - (R_{-a}T_{an+b})^2.$$

Solution by the proposer.

Howard (see (3.6) in [1]) showed that

$$T_{n+2a} = S_aT_{n+a} - S_{-a}T_n + T_{n-a}.$$

Letting $n = ak + b$ in the above identity, we have

$$T_{a(k+2)+b} = S_aT_{a(k+1)+b} - S_{-a}T_{ak+b} + T_{a(k-1)+b}.$$

Let $p = S_a,\ q = S_{-a}$, and $t_n = T_{an+b}$. We have

$$t_{k+2} = pt_{k+1} - t_k + t_{k-1}. \tag{1}$$

We have

$$\begin{aligned}
0 &= 2\sum_{k=1}^{n} t_k((t_{k+2} - pt_{k+1} + qt_k - t_{k-1}) + (t_{k+1} - pt_k + qt_{k-1} - t_{k-2}))\\
&\quad - \sum_{k=1}^{n}((t_{k+1} - pt_k)^2 - (-qt_{k-1} + t_{k-2})^2) \qquad \text{(by (1))}\\
&= (2q - 2p - p^2)\sum_{k=1}^{n} t_k^2 + q^2\sum_{k=1}^{n} t_{k-1}^2 + \sum_{k=1}^{n}(t_{k-2}^2 - t_{k+1}^2)\\
&\quad + 2\sum_{k=1}^{n}(t_k t_{k+1} - t_{k-1}t_k) + 2q\sum_{k=1}^{n}(t_{k-1}t_k - t_{k-2}t_{k-1}) + 2\sum_{k=1}^{n}(t_k t_{k+2} - t_{k-2}t_k)\\
&= (2q - 2p - p^2)\sum_{k=1}^{n} t_k^2 + q^2\left(\sum_{k=1}^{n} t_k^2 - t_n^2 + t_0^2\right) + t_{-1}^2 + t_0^2 + t_1^2 - t_{n-1}^2 - t_n^2 - t_{n+1}^2\\
&\quad + 2(t_n t_{n+1} - t_0 t_1) + 2q(t_{n-1}t_n - t_{-1}t_0) + 2(t_n t_{n+2} + t_{n-1}t_{n+1} - t_{-1}t_1 - t_0 t_2)\\
&= (2q - 2p - p^2 + q^2)\sum_{k=1}^{n} t_k^2 - 2t_0(t_2 + t_1 + qt_{-1}) + (t_1 - t_{-1})^2 + (q^2+1)t_0^2\\
&\quad + 2t_n(t_{n+2} + t_{n+1} + qt_{n-1}) - (t_{n+1} - t_{n-1})^2 - (q^2+1)t_n^2\\
&= ((q+1)^2 - (p+1)^2)\sum_{k=1}^{n} t_k^2 - 2t_0(pt_1 - qt_0 + t_{-1} + t_1 + qt_{-1}) + (t_1 - t_{-1})^2 + (q^2+1)t_0^2\\
&\quad + 2t_n(pt_{n+1} - qt_n + t_{n-1} + t_{n+1} + qt_{n-1}) - (t_{n+1} - t_{n-1})^2 - (q^2+1)t_n^2 \qquad \text{(by (1))}\\
&= (R_{-a}^2 - R_a^2)\sum_{k=1}^{n} t_k^2 - 2t_0((p+1)t_1 + (q+1)t_{-1}) + (t_1 - t_{-1})^2 + (q+1)^2 t_0^2\\
&\quad + 2t_n((p+1)t_{n+1} + (q+1)t_{n-1}) - (t_{n+1} - t_{n-1})^2 - (q+1)^2 t_n^2\\
&= (R_a^2 - R_{-a}^2)\sum_{k=1}^{n} t_k^2 - 2t_0(R_a t_1 + R_{-a}t_{-1}) + (t_1 - t_{-1})^2 + (R_a t_0)^2\\
&\quad + 2t_n(R_a t_{n+1} + R_{-a}t_{-n-1}) - (t_{n+1} - t_{n-1})^2 - (R_{-a}t_n)^2.
\end{aligned}$$

Therefore, we obtain the desired identity.

Note: Using the identity (1), we can also obtain the following identity:

$$(S_a - S_{-a})\sum_{k=1}^{n} T_{ak+b} = T_{an+a+b} + (1 - S_{-a})T_{an+b} - T_{a+b} - (1 - S_{-a})T_b - T_{-a+b}.$$

References

[1] F. T. Howard, *A Tribonacci Identity*, The Fibonacci Quarterly, **39.4** (2001), 352–357.

Also partially solved by Dmitry Fleischman.

Cauchy-Schwartz to the Rescue

H-755 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.
(Vol. 52, No. 1, February 2014)

Let $n \geq 1$ be an integer. Prove that

(1) If $x_k \in \mathbb{R}$ for $k = 1, \ldots, n$, then

$$2\left(\sum_{k=1}^{n} L_k \sin x_k\right)\left(\sum_{k=1}^{n} L_k \cos x_k\right) \leq n(L_n L_{n+1} - 2).$$

(2) If $m \geq 1$, then

$$m^m \sum_{k=1}^{n} (1 + L_{2k-1})^{m+1} \geq (m+1)^{m+1}(L_{2n+2} - 2).$$

Solution to (1) by Adnan Ali, Mumbai, India.

From the AM-GM Inequality and Cauchy-Schwartz Inequality, we have

$$\begin{aligned} 2\left(\sum_{k=1}^{n} L_k \sin x_k\right)\left(\sum_{k=1}^{n} L_k \cos x_k\right) &\leq \left(\sum_{k=1}^{n} L_k \sin x_k\right)^2 + \left(\sum_{k=1}^{n} L_k \cos x_k\right)^2 \\ &\leq \left(\sum_{k=1}^{n} L_k^2\right)\left(\sum_{k=1}^{n} \sin^2 x_k\right) + \left(\sum_{k=1}^{n} L_k^2\right)\left(\sum_{k=1}^{n} \cos^2 x_k\right) \\ &= n\left(\sum_{k=1}^{n} L_k^2\right) = n(L_n L_{n+1} - 2). \end{aligned}$$

Solution to (2) by Ángel Plaza, Gran Canaria, Spain.

Inequality (2) does not hold for some values of m and n (for example, for $m = 1$ and $n = 1, 2$). Instead, we will prove the following modified version:

(2') If $m \geq 1$, then $m^m \sum_{k=1}^{n} (1 + L_{2k-1})^{m+1} \geq (m+1)^{m+1}(L_{2n} - 1)$.

Since $\sum_{k=1}^{n} L_{2k-1} = L_{2n} - 2$, last inequality is equivalent to

$$m^m \sum_{k=1}^{n} \left(\frac{1 + L_{2k-1}}{m+1}\right)^{m+1} \geq \sum_{k=1}^{n} L_{2k-1}.$$

Last inequality follows immediately since function $f(x) = m^m \left(\frac{1+x}{m+1}\right)^{m+1} - x$ is increasing for every $m \geq 1$ and $x \geq 1$, because $f'(x) = m^m \left(\frac{1+x}{m+1}\right)^{m} - 1 \geq 0$ for $x \geq 1$ and $L_{2k-1} \geq 1$ for $k = 1, 2, \ldots, n$.

Also solved by Dmitry Fleischman, Zbigniew Jakubczyk, Hideyuki Ohtsuka, and the proposers.

Note: Concerning **H-688**, the proposer pointed out that the recent references give a negative answer to problem **H-688**.

References

[1] A. Tyszka, *A hypothetical way to compute an upper bound for the heights of solutions of a Diophantine equation with a finite number of solutions*, Annals of Computer Science and Information Systems, **5** (2015), 709–716,

[2] A. Tyszka, *All functions* $g : N \mapsto N$ *which have a single-fold Diophantine representation are dominated by a limit-computable function* $f : N\backslash\{0\} \mapsto N$ *which is implemented in MuPAD and whose computability is an open problem*, Computation, cryptography, and network security (eds. N. J. Daras and M. Th. Rassias), Springer, 2015, 577–590.