ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by October 15, 2007. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1026 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA

Show that the area of a regular pentagon with side s is

$$\frac{s^2(\alpha+2)^{3/2}}{4}.$$

B-1027 Proposed by M. N. Deshpande, Nagpur, India

Define $\{x_n\}$ by $x_1 = 1, x_2 = 10$ and $x_{n+2} = 6x_{n+1} - x_n + 2$ for $n \geq 1$. Let P_n be the n^{th} Pell number for $n \geq 0$.

Prove or disprove:

$$\frac{8x_n(x_n+1)+20}{(P_{2n}-P_{2n-2})^2}$$

is a constant for all integers $n \geq 1$.

B-1028 Proposed by Paul S. Bruckman, Sointula, Canada

Prove that the Diophantine equation

$$(a + b\alpha)^2 + (a + b\beta)^2 = c^2$$

has no solution in the positive integers a, b and c.

B-1029 Proposed by José Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Barcelona, Spain

Let P_n be the n^{th} Pell number. Prove that for all $n \geq 0$,

$$\left\{2\left(P_n^4 + 16P_{n+1}^4 + P_{n+2}^4\right)\right\}^{1/2}$$

is a positive integer.

B-1030 Proposed by H.-J. Seiffert, Berlin, Germany

Let P_n be the n^{th} Pell number.

(a) Let r and s be integers and let $m = \gcd(P_r, F_s, P_{r-1} - F_{s-1})$. Prove that $F_nP_{n+r} \equiv P_nF_{n+s} \pmod{m}$ for all $n \in Z$.

(b) Show that $F_nP_{n+8} \equiv P_nF_{n+18} \pmod{68}$ for all $n \in Z$.

(c) Find integers r and s, not both zero, such that $F_nP_{n+r} \equiv P_nF_{n+s} \pmod{13}$ for all $n \in Z$.

SOLUTIONS

It Follows From Hölder

B-1015 (Correction) **Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universidad Politécnica de Cataluña, Barcelona, Spain (Vol. 44, no. 3, August 2006)**

Let n be a positive integer. Prove that

$$\left(\sum_{k=1}^{n} F_kF_{2k}\right)\left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt{L_k}}\right)^2 \geq F_n^3F_{n+1}^3.$$

Solution by Paul S. Bruckman, P.O. Box 150, Sointula, BC V0N 3E0 (Canada)

We will show that the indicated inequality is a special case of Hölder's Inequality:

$$\sum_{k=1}^{n} a_kb_k \leq \left(\sum_{k=1}^{n}(a_k)^p\right)^{1/p}\left(\sum_{k=1}^{n}(b_k)^q\right)^{1/q}, \tag{1}$$

where p, q, the a_k's and b_k's are positive numbers, with $\frac{1}{p} + \frac{1}{q} = 1$.

Let $p = 3, q = 3/2, a_k = (F_k)^{2/3}(L_k)^{1/3}, B_k = (F_k)^{4/3}(L_k)^{-1/3}$. We see that the indicated quantities satisfy the conditions required of Hölder's Inequality. Then, by (1), we have:

$$\sum_{k=1}^{n}(F_k)^2 \leq \left(\sum_{1}^{n}(F_k)^2 L_k\right)^{1/3}\left(\sum_{k=1}^{n}(F_k)^2(L_k)^{-1/2}\right)^{2/3}.$$

Cubing both sides and noting that $(F_k)^2 L_k = F_k F_{2k}$, we obtain:

$$\left(\sum_{k=1}^{n}(F_k)^2\right)^3 \leq \sum_{k=1}^{n} F_k F_{2k}\left(\sum_{1}^{n}(F_k)^2(L_k)^{-1/2}\right)^2 \qquad (2)$$

Finally, we use the well-known identity:

$$\sum_{k=1}^{n}(F_k)^2 = F_n F_{n+1}. \qquad (3)$$

Substituting the result of (3) into (2) yields the desired inequality.

Also solved by Kenneth Davenport, G. C. Greubel, Harris Kwong, H.-J. Seiffert and the proposer.

A Fibonacci-Lucas Determinant

B-1016 Proposed by Br. J. Mahon, Australia (Vol. 44, no. 2, May 2006)

Prove that

$$\begin{vmatrix} L_{4n+8}+1 & F_{2n+2}F_{2n+4}/3 & 1-F_{2n}^2 \\ L_{4n+4}+1 & F_{2n}F_{2n+2}/3 & 1-F_{2n-2}^2 \\ L_{4n}+1 & F_{2n-2}F_{2n}/3 & 1-F_{2n-4}^2 \end{vmatrix} = 64.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

We shall derive a general result. Define, for any real numbers a, b and c,

$$\Delta = \begin{vmatrix} L_{4n+8}+a & bF_{2n+2}F_{2n+4} & c-F_{2n}^2 \\ L_{4n+4}+a & bF_{2n}F_{2n+2} & c-F_{2n-2}^2 \\ L_{4n}+a & bF_{2n-2}F_{2n} & c-F_{2n-4}^2 \end{vmatrix}.$$

Using the product formula $F_r F_s = (L_{r+s} - (-1)^s L_{r-s})/5$ for $r \geq s$, we can express Δ as

$$\Delta = \frac{b}{25}\begin{vmatrix} L_{4n+8}+a & L_{4n+6}-3 & 5c+2-L_{4n} \\ L_{4n+4}+a & L_{4n+2}-3 & 5c+2-L_{4n-4} \\ L_{4n}+a & L_{4n-2}-3 & 5c+2-L_{4n-8} \end{vmatrix}.$$

Observe that $L_m + L_{m+2} - L_{m+1} = L_{m+2} - (L_{m+3} - L_{m+2}) = -L_{m+3} + 2L_{m+2}$. Continue in this manner, we find $L_m = -8L_{m+7} + 13L_{m+6} = -8L_{m+8} + 21L_m$. Hence

$$-8(L_{m+8} + a) + 21(L_{m+6} - 3) + (5c + 2 - L_m) = -8a + 5c - 61.$$

Also note that $L_{m+4} - L_m = L_{m+4} + L_{m+2} - (L_{m+2} + L_m) = 5(F_{m+3} - F_{m+1}) = 5F_{m+2}$. Therefore

$$\begin{aligned}\Delta &= \frac{b}{25}\begin{vmatrix} L_{4n+8}+a & L_{4n+6}-3 & -8a+5c-61 \\ L_{4n+4}+a & L_{4n+2}-3 & -8a+5c-61 \\ L_{4n}+a & L_{4n-2}-3 & -8a+5c-61 \end{vmatrix} \\ &= \frac{b(-8a+5c-61)}{25}\begin{vmatrix} L_{4n+8}-L_{4n+4} & L_{4n+6}-L_{4n+2} & 0 \\ L_{4n+4}-L_{4n} & L_{4n+2}-L_{4n-2} & 0 \\ L_{4n}+a & L_{4n-2}-3 & 1 \end{vmatrix} \\ &= b(-8a+5c-61)(F_{4n+6}F_{4n} - F_{4n+2}F_{4n+4}) \\ &= b(-8a+5c-61)(L_2 - L_6)/5 \\ &= 3b(8a-5c+61).\end{aligned}$$

In particular, when $a = c = 1$ and $b = 1/3$, we obtain $\Delta = 64$.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Jaroslav Seibert, and the proposer.

A Recurrence Relation For Fibonacci Numbers

B-1017 (Correction) **Proposed by M.N. Deshpande, Nagpur, India (Vol. 44, no. 3, August 2006)**

Define $\{a_n\}$ by $a_1 = a_2 = 0$, $a_3 = a_4 = 1$ and

$$a_n = a_{n-1} + a_{n-3} + a_{n-4} + k(n)$$

for $n \geq 5$ where

$$k(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ i^{n-2} & \text{if } n \text{ is odd} \end{cases}$$

and $i\ \sqrt{-1}$.

Prove or disprove: $a_n + 2a_{n+2} + a_{n+4}$ is a Fibonacci number for all integers $n \geq 1$.

Solution by Russell Jay Hendel, Towson University, Towson, MD

We claim

$$F_{n+3} = a_n + 2a_{n+2} + a_{n+4}, n \geq 1. \tag{1}$$

Clearly (1) holds for $n = 1, 2$. To complete the proof it suffices to show that the right side of (1) obeys the Fibonacci recursion.

First observe that $k(n)$ (mod 4) is the periodic sequence, $0, 1, 0, -1, 0, 1, 0, -1, \ldots$, implying that for all n

$$k(n) = -k(n+2). \tag{2}$$

To show that the right side of (1) obeys the Fibonacci recursion we must show

$$(a_n + 2a_{n+2} + a_{n+4}) + (a_{n+1} + 2a_{n+3} + a_{n+5}) = (a_{n+2} + 2a_{n+4} + a_{n+6}),$$

or equivalantly, gathering similar terms and cancelling, we must prove

$$\begin{aligned} a_{n+6} &= a_{n+5} - a_{n+4} + 2a_{n+3} + a_{n+2} + a_{n+1} + a_n \\ &= a_{n+5} + a_{n+3} + a_{n+2} - k(n+4), \text{by the defining recursion for } a_{n+4}, \\ &= a_{n+5} + a_{n+3} + a_{n+2} + k(n+6), \text{ by (2).} \end{aligned}$$

But this last equation is identical to the defining recursion for a_{n+6} completing the proof.

Also solved by Paul S. Bruckman, Charles K. Cook, Kenneth Davenport, G.C. Greubel, Harris Kwong, Graham Lord, H.-J. Seiffert, and the proposer.

It Only Looks Complicated!

B-1018 Proposed by Mohammad K. Azarian, Evansville, Indiana (Vol. 44, no. 2, May 2006)

If $n \geq 2$ and $\binom{n}{k}$ denotes the binomial coefficient, show that

$$\begin{aligned} f(F,L) =& F_{n+2} - (F_2 + F_4 + F_6 + \cdots + F_{2n} - F_{2n+1})^{F_{n+1}} \binom{L_n}{F_{n+1}} \binom{L_n}{F_{n-1}} \binom{-F_{n-1} - 1}{F_{n+1}} \\ &+ (F_2 + F_4 + F_6 + \cdots + F_{2n} - F_{2n+1})^{1+F_{n+1}} \binom{L_n}{F_{n+1}} \binom{-F_{n-1} - 1}{F_{n+1}} \\ &+ (F_2 + F_4 + F_6 + \cdots + F_{2n} - F_{2n+1})^{1+F_{n+1}} \binom{-F_{n-1} - 1}{F_{n+1}} \\ &- (F_1 + F_2 + F_3 + \cdots + F_n) \end{aligned}$$

can be written in the form MN^2, where M and N are natural numbers.

Solution by Kenneth Davenport, AF-7291, Dallas, PA

We must show that $f(F,L)$ may be written in the form MN^2, where M and N are natural numbers.

First, we simplify each of the series given in the braces; namely, it is known that

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

and

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1.$$

Further, we make use of the binomial identity

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

Thus $f(F,L)$ is simplified as follows:

$$\begin{aligned}
&= F_{n+2} + (-1)^{F_{n+1}}(-1)^{F_{n+1}}\binom{L_n}{F_{n+1}}\binom{L_n}{F_{n-1}}\binom{F_{n+1}+F_{n-1}}{F_{n+1}} \\
&+ (-1)^{1+F_{n+1}}(-1)^{F_{n+1}}\binom{L_n}{F_{n+1}}\binom{F_{n+1}+F_{n-1}}{F_{n+1}} \\
&+ (-1)^{1+F_{n+1}}(-1)^{F_{n+1}}\binom{F_{n+1}+F_{n-1}}{F_{n+1}} \\
&- (F_{n+2}-1)\,. \qquad (1)
\end{aligned}$$

But, $F_{n+1}+F_{n-1}=L_n$, therefore (1) becomes:

$$1-\binom{L_n}{F_{n+1}}-\binom{L_n}{F_{n+1}}^2+\binom{L_n}{F_{n-1}}\binom{L_n}{F_{n+1}}^2. \qquad (2)$$

Now it is clear $\binom{L_n}{F_{n+1}}$ and $\binom{L_n}{F_{n-1}}$ are equivalent by virtue of $F_{n+1}+F_{n-1}=L_n$. Therefore (2) is a cubic of the form:

$$(1-x-x^2+x^3) \text{ which factors as } (x-1)^2(x+1).$$

Letting $M=\binom{L_n}{F_{n+1}}-1$ and $N=\binom{L_n}{F_{n+1}}+1$ we get the desired relation.

Also solved by Paul S. Bruckman, Russell J. Hendel, Harris Kwong, Jaroslav Seibert, H.-J. Seiffert, and the proposer.