ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-805 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that if $n \geq 2$, $p \geq 1$ are integers and $m \geq 0$, $x_k > 0$ are real numbers for $k = 1, \ldots, n$, then letting $X_n = \sum_{k=1}^{n} x_k$, we have the inequality

$$\sum_{k=1}^{n} \frac{(F_p X_n + F_{p+1} x_k)^{m+1}}{(F_{p+1}^2 X_n - F_p^2 x_k)^{2m+1}} \geq \frac{(nF_p + F_{p+1})^{m+1} n^{m+1}}{(nF_{p+1}^2 - F_p^2)^{2m+1} X_n^m}.$$

H-806 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

The two sequences $\{T_n\}_{n\in\mathbb{Z}}$ and $\{S_n\}_{n\in\mathbb{Z}}$ satisfy

$$\begin{aligned} T_{n+3} &= T_{n+2} + T_{n+1} + T_n \qquad \text{with} \qquad T_0 = 0,\ T_1 = T_2 = 1, \\ S_{n+3} &= S_{n+2} + S_{n+1} + S_n \qquad \text{with} \qquad S_0 = 3,\ S_1 = 1,\ S_2 = 3 \end{aligned}$$

for all integers n. For $n \geq 0$ prove that

$$\sum_{k=0}^{n} T_{(-2)^k} S_{(-2)^k} = T_{2(-2)^n}.$$

H-807 Proposed by Mehtaab Sawhney, Commack, NY.

Prove for positive integers n that

$$\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor \sum_{j=1}^{i} \mu(\gcd(i,j)) = \sum_{k=1}^{n} \phi(k),$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mu(\gcd(i,j)) \left\lfloor \sqrt{\frac{n}{ij}} \right\rfloor = \sum_{k=1}^{n} 2^{\omega(k)}.$$

H-808 **Proposed by Mehtaab Sawhney, Commack, NY.**

Prove that

$$\sum_{j=0}^{\lfloor n/2\rfloor} \binom{n}{j, j, n-2j} = \sum_{i=0}^{\lfloor n/2\rfloor} (-1)^i \binom{n}{i} \binom{2n-1-3i}{n-1}.$$

SOLUTIONS

An Integral with the Gamma Function and Fibonacci Numbers

H-771 **Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.**
(Vol. 53, No. 2, May 2015)

Let $m > 0$ and $\Gamma : (0,\infty) \to (0,\infty)$ be the gamma function. Calculate

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} \Gamma\left(\frac{x}{n}\sqrt[n]{F_n}^m\right) dx.$$

Solution by Ángel Plaza.

We will show that f is a continuous real function in (a,b) and $\alpha^m/e \in (a,b)$ then

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\sqrt[n]{F_n}^m\right)\, dx = \frac{1}{e} f\left(\frac{\alpha^m}{e}\right).$$

In our case, Γ is a continuous real function in $(0,\infty)$ and therefore the required limit is $\frac{1}{e}\Gamma\left(\frac{\alpha^m}{e}\right)$.

Let $b_n = \sqrt[n+1]{(n+1)!}\frac{\sqrt[n]{F_n}^m}{n}$ and $a_n = \sqrt[n]{n!}\frac{\sqrt[n]{F_n}^m}{n}$. Then, by the Mean Value Theorem for integrals,

$$\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\sqrt[n]{F_n}^m\right)\, dx = \frac{n}{\sqrt[n]{F_n}^m}\int_{a_n}^{b_n} f(t)\, dt = \frac{n}{\sqrt[n]{F_n}^m}(b_n - a_n) f(t_n)$$

for some $t_n \in (a_n, b_n)$. Now, by the Stirling approximation formula,

$$\ln(n!) = n\ln(n) - n + \frac{1}{2}\ln(n) + \ln(\sqrt{2\pi}) + O\left(\frac{1}{n}\right),$$

so

$$\ln\left(\frac{\sqrt[n]{n!}}{n}\right) = \frac{\ln n!}{n} - \ln n = -1 + O\left(\frac{\ln n}{n}\right) = -1 + o(1)$$

as $n \to \infty$. Thus, using also the Binet formula for F_n which implies that $\lim_{n\to\infty} \sqrt[n]{F_n} = \alpha$, we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} t_n = \frac{\alpha^m}{e}.$$

By the continuity of f at α^m/e, we have

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{F_n}^m}(b_n - a_n) f(t_n) = f\left(\frac{\alpha^m}{e}\right) \lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)$$
$$= f\left(\frac{\alpha^m}{e}\right) \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} f\left(\frac{\alpha^m}{e}\right).$$

Also solved by Dmitry Fleischman, Nicuşor Zlota, and the proposers.

A Geometric Inequality

H-772 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.
(Vol. 53, No. 2, May 2015)

If ABC is a noniscosceles triangle then prove that

$$\sum_{\substack{cyclic\\permutations}} \frac{a^8}{(bF_n^2 + cF_{n+1}^2)(a-b)^2(a-c)^2} > \frac{288r^3\sqrt{3}}{F_{2n+1}}.$$

Here, a, b, c, r are the lengths of the sides and the radius of the inscribed circle of the triangle ABC, respectively.

Solution by the proposers.

By the Harald Bergström inequality and $F_n^2 + F_{n+1}^2 = F_{2n+1}$, we have:

$$W = \sum_{\substack{cyclic\\permutations}} \frac{a^8}{(bF_n^2 + cF_{n+1}^2)(a-b)^2(a-c)^2}$$
$$= \sum_{\substack{cyclic\\permutations}} \frac{\left(\frac{a^4}{(a-b)(a-c)}\right)^2}{bF_n^2 + cF_{n+1}^2} \geq \frac{\left(\sum_{\substack{cyclic\\permutations}} \frac{a^4}{(a-b)(a-c)}\right)^2}{\sum_{\substack{cyclic\\permutations}} (bF_n^2 + cF_{n+1}^2)}$$
$$= \frac{1}{(a+b+c)(F_n^2 + F_{n+1}^2)} \left(\sum_{\substack{cyclic\\permutations}} \frac{a^4}{(a-b)(a-c)}\right)^2$$
$$= \frac{1}{(a+b+c)F_{2n+1}} \left(\sum_{\substack{cyclic\\permutations}} \frac{a^4}{(a-b)(a-c)}\right)^2.$$

The sum in parentheses simplifies to

$$\sum_{\substack{cyclic\\permutations}} \frac{a^4}{(a-b)(a-c)} = \frac{-a^4(b-c) - b^4(c-a) - c^4(a-b)}{(a-b)(b-c)(c-a)}$$
$$= a^2 + b^2 + c^2 + ab + bc + ca.$$

Since $a^2+b^2+c^2 \geq ab+bc+ca \geq 4S\sqrt{3}$, we get

$$W \geq \frac{1}{(a+b+c)F_{2n+1}}(8S\sqrt{3})^2 = \frac{192S^2}{2pF_{2n+1}} = \frac{192(pr)^2}{2pF_{2n+1}} = \frac{96pr^2}{F_{2n+1}} \geq \frac{288r^3\sqrt{3}}{F_{2n+1}},$$

where for the last inequality we used the fact that $p \geq 3\sqrt{3}r$.

Remark. The inequality is strict because ABC is not equilateral.

A Sum with Binomial Coefficients, Fibonacci and Bernoulli Numbers

H-773 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 53, No. 3, August 2015)

Let B_n be the Bernoulli numbers defined by the generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n.$$

For integers $n \geq 0$ and $m \geq 0$, prove that

$$\sum_{k=0}^{n} \binom{2n}{2k} F_{2mk}B_{2(n-k)} = \frac{n}{\sqrt{5}}\left[2\sum_{r=1}^{L_m}(\alpha^m - r)^{2n-1} + L_{m(2n-1)}\right].$$

Solution by the proposer.

It is known that

$$B_n(x+1) - B_n(x) = nx^{n-1}, \qquad \text{where} \qquad B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k.$$

By this identity, we have

$$\sum_{k=0}^{2n}\binom{2n}{k}((\alpha^m - r+1)^k - (\alpha^m - r)^k)B_{2n-k} = 2n(\alpha^m - r)^{2n-1}.$$

Using this identity, we have

$$\begin{aligned}
\sum_{r=1}^{L_m} 2n(\alpha^m - r)^{2n-1} &= \sum_{k=0}^{2n}\binom{2n}{k}\left\{\sum_{r=1}^{L_m}((\alpha^m - r+1)^k - (\alpha^m - r)^k)\right\}B_{2n-k} \\
&= \sum_{k=0}^{2n}\binom{2n}{k}(\alpha^{mk} - (\alpha^m - L_m)^k)B_{2n-k} = \sum_{k=0}^{2n}\binom{2n}{k}(\alpha^{mk} - (-\beta^m)^k)B_{2n-k} \\
&= \sum_{k=0}^{n}\binom{2n}{2k}(\alpha^{2mk} - \beta^{2mk})B_{2(n-k)} + \binom{2n}{2n-1}(\alpha^{m(2n-1)} + \beta^{m(2n-1)})B_1 \\
&= \sqrt{5}\sum_{k=0}^{n}\binom{2n}{2k}F_{2mk}B_{2(n-k)} - nL_{m(2n-1)}.
\end{aligned}$$

Therefore, we obtain the desired identity.

Also solved by Dmitry Fleischman.

Bessel Functions with Fibonacci and Lucas Numbers

H-774 Proposed by G. C. Greubel, Newport News, VA.
(Vol. 53, No. 3, August 2015)

1. Let $m \geq 0,\ p \geq 0$ be integers. Evaluate the series
$$\sum_{n=0}^{\infty} \frac{F_{n+p}L_{n+m}}{(n+p)!(n+m)!}$$
in terms of the Bessel functions.
2. Evaluate the case $m = p$ in terms of a series of modified Bessel functions of the first kind. Take the limiting case $m \to 0$.
3. Show that when $p = 0$ the series is given by
$$\sum_{n=0}^{\infty} \frac{F_n L_{n+m}}{n!(n+m)!} = \frac{1}{\sqrt{5}} \left(I_m(2\alpha) - I_m(2\beta) - F_m J_m(2) \right).$$

Solution by the proposer.

Part 1

Let the series in question be given by
$$S_p^m = \sum_{n=0}^{\infty} \frac{F_{n+p}L_{n+m}}{(n+p)!(n+m)!}.$$
Without much difficulty it is seen that
$$F_{n+p}L_{n+p} = F_{2n+p+m} + (-1)^{n+m}F_{p-m}.$$
Use of this expression leads the series S_p^m to the form
$$S_p^m = \sum_{n=0}^{\infty} \frac{F_{2n+p+m}}{(n+p)!(n+m)!} + (-1)^m F_{p-m} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+p)!(n+m)!}.$$
This current expression can be more easily seen in the form
$$S_p^m = \frac{1}{\sqrt{5}m!p!} \left(\alpha^{p+m} f(\alpha^2; p, m) - \beta^{p+m} f(\beta^2; p, m) \right) + \frac{(-1)^m F_{p-m}}{m!p!} f(-1; p, m), \tag{1}$$
where
$$f(x; p, m) = \sum_{n=0}^{\infty} \frac{x^n}{(p+1)_n (m+1)_n}. \tag{2}$$
The series given by $f(x; p, m)$ is of the hypergeometric type ${}_1F_2$ and can then be related to the Lommel functions, which are of the Bessel "family" of functions. The Lommel functions are expressed by
$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\ {}_1\mathrm{F}_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right).$$

When μ and ν are set to the values $\mu = p+m-1$ and $\nu = m-p$ the Lommel function reduces to

$$s_{m+p-1,m-p}(z) = \frac{z^{m+p}}{4mp}\ {}_1\mathrm{F}_2\left(1; p+1, m+1; -\frac{z^2}{4}\right).$$

Upon making the change of variable $z = 2i\sqrt{x}$ it is seen that

$$s_{m+p-1,m-p}(2i\sqrt{x}) = \frac{2^{m+p-2} i^{m+p} x^{(m+p)/2}}{mp}\ {}_1F_2(1; p+1, m+1; x). \tag{3}$$

Comparison of equations (2) and (3) lead to

$$f(x; p, m) = (mp)\ \frac{2^{2-m-p}(-i)^{m+p}}{x^{(m+p)/2}}\ s_{m+p-1,m-p}(2i\sqrt{x}).$$

With this result equation (1) becomes

$$\begin{aligned} S_p^m = &\frac{(-i)^{m+p}\ 2^{2-m-p}}{\sqrt{5}\Gamma(m)\Gamma(p)} \left[s_{m+p-1,m-p}(2i\alpha) - s_{m+p-1,m-p}(2i\beta)\right] \\ &+ \frac{(-1)^p\ 2^{2-m-p} F_{p-m}}{\Gamma(m)\Gamma(p)}\ s_{m+p-1,m-p}(-2). \end{aligned} \tag{4}$$

As an alternate form the modified Lommel functions can be used, given by (see paper [1] and the references therein):

$$t_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\ {}_1\mathrm{F}_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; \frac{z^2}{4}\right),$$

and have the relation $t_{\mu,\nu}(x) = (-i)^{\mu+1} s_{\mu,\nu}(ix)$. With this, equation (4) becomes

$$\begin{aligned} S_p^m = &\frac{2^{2-m-p}}{\sqrt{5}\ \Gamma(m)\Gamma(p)} \left[t_{m+p-1,m-p}(2\alpha) - t_{m+p-1,m-p}(2\beta)\right] \\ &+ \frac{(-1)^p\ 2^{2-m-p} F_{p-m}}{\Gamma(m)\Gamma(p)}\ s_{m+p-1,m-p}(-2). \end{aligned}$$

The desired relation sought is, or equation (4),

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{F_{n+p} L_{n+m}}{(n+p)!(n+m)!} = &\frac{2^{2-m-p}}{\sqrt{5}\ \Gamma(m)\Gamma(p)} \left[t_{m+p-1,m-p}(2\alpha) - t_{m+p-1,m-p}(2\beta)\right] \\ &+ \frac{(-1)^p\ 2^{2-m-p} F_{p-m}}{\Gamma(m)\Gamma(p)}\ s_{m+p-1,m-p}(-2). \end{aligned}$$

Part 2

Lommel's function can be expanded in terms of a series involving the Bessel function of the first kind. When $\mu \pm \nu \neq -1, -2, \ldots$ it is given that (see equation 11.9.7 in [2]):

$$s_{\mu,\nu}(z) = 2^{\mu+1} \sum_{k=0}^{\infty} \frac{(2k+\mu+1)\Gamma(k+\mu+1)}{k!(2k+\mu-\nu+1)(2k+\mu+\nu+1)}\ J_{2k+\mu+1}(z).$$

When $z = ix$, the Bessel function becomes the modified Bessel function of the first kind and is given by $J_m(ix) = i^m I_m(x)$, the result is

$$s_{\mu,\nu}(ix) = (2i)^{\mu+1} \sum_{k=0}^{\infty} \frac{(-1)^k (2k+\mu+1)\Gamma(k+\mu+1)}{k!(2k+\mu-\nu+1)(2k+\mu+\nu+1)}\ I_{2k+\mu+1}(z).$$

When $\mu = p + m - 1$ and $\nu = m - p$ this becomes

$$s_{m+p-1,m-p}(ix) = -\sum_{k=0}^{\infty} \frac{(2i)^{m+p-2}(-1)^k(2k+m+p)\Gamma(k+m+p)}{k!(k+p)(k+m)} \cdot I_{2k+m+p}(x).$$

Making use of this relation equation (4) becomes

$$S_p^m = \frac{1}{\sqrt{5}\ \Gamma(m)\Gamma(p)} \sum_{k=0}^{\infty} \frac{(-1)^k(2k+m+p)\Gamma(k+m+p)}{k!(k+p)(k+m)} \cdot \left[I_{2k+m+p}(2\alpha) - I_{2k+m+p}(2\beta) + \sqrt{5}\ (-1)^{k+p} F_{p-m} J_{2k+m+p}(-2)\right].$$

When $m = p$ this reduces to

$$\sum_{n=0}^{\infty} \frac{F_{2n+2m}}{[(n+m)!]^2} = \frac{2}{\sqrt{5}\ \Gamma^2(m)} \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(k+2m)}{k!(k+m)} \cdot [I_{2k+2m}(2\alpha) - I_{2k+2m}(2\beta)]$$

or

$$\sum_{n=0}^{\infty} \frac{F_{2n+2m}}{[(n+m)!]^2} = \frac{m}{\sqrt{5}} \binom{2m}{m} \sum_{k=0}^{\infty} \frac{(-1)^k(2m)_k}{k!(k+m)} \left[I_{2k+2m}(2\alpha) - I_{2k+2m}(2\beta)\right].$$

This is the desired result of Part 2. It may be noted than when $m \to 0$ the expression can be reduced to

$$\sum_{n=0}^{\infty} \frac{F_{2n}}{[(n)!]^2} = \frac{1}{\sqrt{5}} \left[I_0(2\alpha) - I_0(2\beta)\right]. \tag{5}$$

Part 3

Since $F_n L_{n+m} = F_{2n+m} - (-1)^n F_m$ it can be easily seen that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{F_n L_{n+m}}{n!(n+m)!} &= \sum_{n=0}^{\infty} \frac{F_{2n+m}}{n!(n+m)!} - F_m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \\ &= \frac{1}{\sqrt{5}} \left[\alpha^m \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!(n+m)!} - \beta^m \sum_{n=0}^{\infty} \frac{\beta^{2n}}{n!(n+m)!}\right] \\ &\quad - F_m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!}\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{F_n L_{n+m}}{n!(n+m)!} = \frac{1}{\sqrt{5}} \left(I_m(2\alpha) - I_m(2\beta)\right) - F_m J_m(2),$$

where $J_m(x)$ and $I_m(x)$ are the Bessel and modified Bessel functions of the first kind, respectively. When $m = 0$ this result reproduces (5).

From the relation $F_{n+p} L_n = F_{2n+p} + (-1)^p F_p$ it follows that

$$\sum_{n=0}^{\infty} \frac{F_{n+p} L_n}{n!(n+p)!} = \frac{1}{\sqrt{5}} \left(I_p(2\alpha) - I_p(2\beta)\right) + F_p J_p(2).$$

References

[1] C. H. Zeiner and H. P. Schlemmer, *The inverse Laplace transforms of the modified Lommel functions*, Integral Transforms and Special Functions, **24.2** (2013), 141–155.

[2] *Digital Library of Mathematical Functions*, DLMF, `http://dlmf.nist.gov/11.9.`

Also solved by Dmitry Fleischman.