ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany, or by email at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-926 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $G_n = \alpha^n + \alpha^{-n}$. For integers r and s, prove that

$$\prod_{n=1}^{\infty} \frac{L_{2n} + L_{2r}}{L_{2n} + L_{2s}} = \alpha^{r^2 - s^2} \frac{G_s}{G_r}.$$

H-927 Proposed by Florică Anastase, Lehliu-Gară, Romania

Prove that, for all $n \geq 2$,

$$\frac{L_2}{(2L_1+L_2)^2} + \frac{L_3}{(2L_1+2L_2+L_3)^2} + \cdots + \frac{L_n}{(2L_1+2L_2+\cdots+2L_{n-1}+L_n)^2} \leq \frac{L_{n+2}-L_3}{L_1L_3(L_{n+2}-L_2)}.$$

H-928 Proposed by Ángel Plaza, Gran Canaria, Spain

Prove that, for nonnegative integers m,

$$\sum_{j=0}^{m} \binom{2m+1}{m-j} F_{2j+1}(2j+1) = \sum_{j=0}^{m} \binom{2m-2j}{m-j} 5^{j-1}(5-2j).$$

H-929 Proposed by Toyesh Prakash Sharma, Agra, India

For $n \geq 2$, show that

$$\frac{F_n\alpha^{F_n} + L_n\alpha^{L_n}}{2} \geq \frac{\alpha^{L_n}(L_n \ln\alpha - 1) - \alpha^{F_n}(F_n \ln\alpha - 1)}{L_n \ln^2\alpha - F_n \ln^2\alpha} \geq F_{n+1}\alpha^{F_{n+1}}.$$

H-930 Proposed by the editor

Let G_n be F_n or L_n. For $m \geq 0$ and $n \geq 1$, prove the following identities.

$$\sum_{k=1}^{n} \binom{n}{k} G_{k+m-1} H_k = \sum_{k=1}^{n} G_{2n+m-2k}(H_n - H_{k-1}),$$

$$\sum_{k=1}^{n} \binom{n}{k} G_{3k+m-3} H_k = \sum_{k=1}^{n} 2^{n-k} G_{2n+m-2k}(H_n - H_{k-1}),$$

$$\sum_{k=1}^{n}\binom{n}{k}2^{k-1}G_{k+m-1}H_k=\sum_{k=1}^{n}G_{3n+m-3k}(H_n-H_{k-1}),$$

$$\sum_{k=1}^{n}\binom{n}{k}3^{k-1}G_{3k+m-3}H_k=\sum_{k=1}^{n}2^{n-k}G_{4n+m-4k}(H_n-H_{k-1}),$$

where $H_n=\sum_{m=1}^{n}1/m$, $H_0=0$, is the nth harmonic number.

SOLUTIONS

H-895 Proposed by Andrei K. Svinin, Irkutsk, Russia
(Vol. 60, No. 2, May 2022)

Consider the Genocchi numbers $G_{2n}=(-1)^{n-1}2(4^n-1)B_{2n}$ for $n\geq 1$, where B_{2n} is the Bernoulli number.

(1) Prove that $\displaystyle\sum_{j=0}^{\lfloor (n-1)/3\rfloor}\frac{1}{2j+1}\binom{n-j-1}{2j}\left(\frac{4}{27}\right)^j=\frac{4^n-1}{3^{n-1}(2n+1)}$ and deduce that

$$G_{2n}=\sum_{j=0}^{\lfloor (n-1)/3\rfloor}G_{2n}^{(j)},\text{ where } G_{2n}^{(j)}=(-1)^{n-1}\frac{2^{2j+1}}{3^{3j-n+1}}\frac{2n+1}{2j+1}\binom{n-j-1}{2j}B_{2n}.$$

(2) Show that $G_{2p}^{(j)}\in\mathbb{N}$ for all $j=0,1,\ldots,\lfloor (p-1)/3\rfloor$ if and only if p is prime.

(3) Prove that the g.c.d. of the set of numbers $\{G_{2p}^{(j)}:j=0,\ldots,\lfloor (p-1)/3\rfloor\}$ with a fixed prime $p\geq 5$ is the numerator of the Bernoulli number B_{2p}.

No complete solution was submitted for this problem proposal. The problem remains open. A partial solution was submitted by Dmitry Fleischman. Also, the proposer informed the editor that in part (2) the "if and only if" statement is wrong. A counterexample is $p=49$.

H-896 Proposed by Mihály Bencze, Braşov, Romania
(Vol. 60, No. 2, May 2022)

Prove that

(1) $n\sum_{k=1}^{n}F_k^3+(F_{n+2}-1)^3\leq (n+1)F_nF_{n+1}(F_{n+2}-1)$ holds for all $n\geq 1$;

(2) $n\sum_{k=1}^{n}L_k^3+(L_{n+2}-1)^3\leq (n+1)(L_nL_{n+1}-2)(L_{n+2}-1)$ holds for all $n\geq 1$.

Solution by Michel Bataille, Rouen, France

Inequality (2) does not hold for $n=1,2$. Instead of (2), we will prove

$$(2)'\qquad n\sum_{k=1}^{n}L_k^3+(L_{n+2}-3)^3\leq (n+1)(L_nL_{n+1}-2)(L_{n+2}-3),$$

which was likely the intended inequality.

For all positive integers n, the following well-known formulas hold (and are easily proved by induction).

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1, \quad \sum_{k=1}^{n} F_k^2 = F_n F_{n+1}, \quad \sum_{k=1}^{n} L_k = L_{n+2} - 3, \quad \sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2.$$

As a consequence, inequalities (1) and (2)$'$ are the particular cases $x_k = F_k$ and $x_k = L_k$, respectively, of the following general result: If $x_1, x_2, \ldots, x_n$ are positive real numbers, then

$$n \sum_{k=1}^{n} x_k^3 + \left(\sum_{k=1}^{n} x_k \right)^3 \leq (n+1) \left(\sum_{k=1}^{n} x_k^2 \right) \left(\sum_{k=1}^{n} x_k \right).$$

Proof. Equality holds if $n = 1$, so we assume that $n \geq 2$. Let $S_m = \sum_{k=1}^{n} x_k^m$. Due to homogeneity, we have to prove

$$nS_3 + 1 \leq (n+1)S_2, \tag{3}$$

given that $S_1 = 1$.

We have $(n+1)S_2 - nS_3 = S_2 + n(S_2 S_1 - S_3) = S_2 + n \cdot \sum_{k=1}^{n} x_k (S_2 - x_k^2)$.

The Cauchy-Schwarz inequality gives

$$(n-1)(S_2 - x_k^2) \geq (S_1 - x_k)^2 = (1 - x_k)^2.$$

Hence,

$$\begin{aligned} (n+1)S_2 - nS_3 \geq S_2 + \frac{n}{n-1} \sum_{k=1}^{n} x_k (1 - x_k)^2 &= S_2 + \frac{n}{n-1}(S_1 - 2S_2 + S_3) \\ &= \frac{n}{n-1} - \frac{(n+1)S_2 - nS_3}{n-1} \end{aligned}$$

and (3) readily follows.

Also solved by Brian Bradie, Dmitry Fleischman, Ángel Plaza, Albert Stadler, Andrés Ventas, and the proposer.

Editor's remark: Brian Bradie and Ángel Plaza explicitly mentioned the connection to Muirhead's inequality.

<u>H-897</u> Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 60, No. 2, May 2022)

Prove that

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{L_{2F_n} L_{2F_{n+1}} L_{2F_{n+2}}} = \sum_{n=0}^{\infty} \frac{1}{L_{2F_{2n}} L_{2F_{2n+3}}};$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{L_{F_n}^2 L_{F_{n+1}} L_{F_{n+2}} L_{F_{n+3}}} = \sum_{n=0}^{\infty} \frac{1}{L_{F_n}^2 L_{F_{n+3}}^2} + \frac{1}{4}.$

Solution by Won Kyun Jeong, Daegu, South Korea

For (i), it follows from the identity $L_s L_t = L_{s+t} + (-1)^s L_{t-s}$ that we have

$$L_{2F_{n+1}} L_{2F_{n+2}} = L_{2F_{n+3}} + L_{2F_n}.$$

Then, we obtain

$$\frac{1}{L_{2F_n}L_{2F_{n+3}}} = \frac{L_{2F_{n+3}} + L_{2F_n}}{L_{2F_n}L_{2F_{n+1}}L_{2F_{n+2}}L_{2F_{n+3}}} = \frac{1}{L_{2F_n}L_{2F_{n+1}}L_{2F_{n+2}}} + \frac{1}{L_{2F_{n+1}}L_{2F_{n+2}}L_{2F_{n+3}}}.$$

Because

$$\frac{1}{L_{2F_{2n}}L_{2F_{2n+3}}} = \frac{1}{L_{2F_{2n}}L_{2F_{2n+1}}L_{2F_{2n+2}}} + \frac{1}{L_{2F_{2n+1}}L_{2F_{2n+2}}L_{2F_{2n+3}}},$$

we find that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{L_{2F_{2n}}L_{2F_{2n+3}}} &= \sum_{n=0}^{\infty}\left(\frac{1}{L_{2F_{2n}}L_{2F_{2n+1}}L_{2F_{2n+2}}} + \frac{1}{L_{2F_{2n+1}}L_{2F_{2n+2}}L_{2F_{2n+3}}}\right)\\
&= \left(\frac{1}{L_{2F_0}L_{2F_1}L_{2F_2}} + \frac{1}{L_{2F_1}L_{2F_2}L_{2F_3}}\right) + \left(\frac{1}{L_{2F_2}L_{2F_3}L_{2F_4}} + \frac{1}{L_{2F_3}L_{2F_4}L_{2F_5}}\right) + \cdots\\
&= \sum_{n=0}^{\infty}\frac{1}{L_{2F_n}L_{2F_{n+1}}L_{2F_{n+2}}}.
\end{aligned}$$

This proves (i). Now we prove (ii). Note that it may be written as

$$\sum_{n=0}^{\infty}\left(\frac{2}{L_{F_n}^2 L_{F_{n+1}}L_{F_{n+2}}L_{F_{n+3}}} - \frac{1}{L_{F_n}^2 L_{F_{n+3}}^2}\right) = \frac{1}{4}.$$

Because

$$2L_{F_{n+3}} - L_{F_{n+1}}L_{F_{n+2}} = 5F_{F_{n+1}}F_{F_{n+2}},$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\frac{2}{L_{F_n}^2 L_{F_{n+1}}L_{F_{n+2}}L_{F_{n+3}}} - \frac{1}{L_{F_n}^2 L_{F_{n+3}}^2}\right) &= \sum_{n=0}^{\infty}\frac{2L_{F_{n+3}} - L_{F_{n+1}}L_{F_{n+2}}}{L_{F_n}^2 L_{F_{n+1}}L_{F_{n+2}}L_{F_{n+3}}^2}\\
&= \sum_{n=0}^{\infty}\frac{5F_{F_{n+1}}F_{F_{n+2}}}{L_{F_n}^2 L_{F_{n+1}}L_{F_{n+2}}L_{F_{n+3}}^2}.
\end{aligned}$$

Using the identity

$$L_{m+n}^2 - L_{m-n}^2 = 5F_{2m}F_{2n},$$

we find that

$$L_{F_{n+3}}^2 - L_{F_n}^2 = 5F_{2F_{n+1}}F_{2F_{n+2}} = 5F_{F_{n+1}}L_{F_{n+1}}F_{F_{n+2}}L_{F_{n+2}}.$$

Finally, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{2}{L_{F_n}^2L_{F_{n+1}}L_{F_{n+2}}L_{F_{n+3}}}-\frac{1}{L_{F_n}^2L_{F_{n+3}}^2}\right)\\
&\quad= \sum_{n=0}^{\infty}\frac{5F_{F_{n+1}}F_{F_{n+2}}}{L_{F_{n+3}}^2-L_{F_n}^2}\left(\frac{1}{L_{F_n}^2L_{F_{n+1}}L_{F_{n+2}}}-\frac{1}{L_{F_{n+1}}L_{F_{n+2}}L_{F_{n+3}}^2}\right)\\
&\quad= \sum_{n=0}^{\infty}\frac{1}{L_{F_{n+1}}L_{F_{n+2}}}\left(\frac{1}{L_{F_n}^2L_{F_{n+1}}L_{F_{n+2}}}-\frac{1}{L_{F_{n+1}}L_{F_{n+2}}L_{F_{n+3}}^2}\right)\\
&\quad= \sum_{n=0}^{\infty}\left(\frac{1}{L_{F_n}^2L_{F_{n+1}}^2L_{F_{n+2}}^2}-\frac{1}{L_{F_{n+1}}^2L_{F_{n+2}}^2L_{F_{n+3}}^2}\right)\\
&\quad= \frac{1}{L_{F_0}^2L_{F_1}^2L_{F_2}^2}\\
&\quad= \frac{1}{4}.
\end{aligned}$$

This completes the proof.

Also solved by Dmitry Fleischman, Ángel Plaza, and the proposer.

H-898 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania
(Vol. 60, No. 2, May 2022)

Compute

$$\lim_{n\to\infty}(\sqrt[n]{n!})^2\left(\frac{\sqrt[n]{n!L_n}}{n^2}-\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2}\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland

We will prove that the limit equals $\alpha\frac{1+\ln\sqrt{5}}{e^3}$.
We use Stirling's asymptotic formula for factorials in the form

$$n!=\sqrt{2\pi n}n^ne^{-n+O\left(\frac{1}{n}\right)},\quad n\to\infty.$$

Hence,

$$\sqrt[n]{n!}=\frac{n}{e}+\frac{1}{2e}\ln(2\pi n)\ +O\left(\frac{\ln^2 n}{n}\right)$$

and

$$\left(\sqrt[n]{n!}\right)^2=\frac{1}{e^2}n^2+\frac{1}{e^2}\ln(2\pi n)\ n+O\left(\ln^2 n\ \right).$$

Furthermore,

$$\sqrt[n]{L_n}=\alpha\left(1+\left(-\frac{1}{\alpha^2}\right)^n\right)^{\frac{1}{n}}=\alpha+O\left(\frac{1}{n\alpha^{2n}}\right),$$

$$\sqrt[n]{F_n}=\frac{1}{\sqrt[2n]{5}}\alpha\left(1-\left(-\frac{1}{\alpha^2}\right)^n\right)^{\frac{1}{n}}=\alpha\left(1-\frac{\ln 5}{2n}\right)+O\left(\frac{1}{n^2}\right).$$

We collect results and find that

$$
\begin{aligned}
&(\sqrt[n]{n!})^2\Big(\frac{\sqrt[n]{n!L_n}}{n^2}-\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2}\Big)\\
&=\Big(\frac{1}{e^2}n^2+\frac{1}{e^2}\ln(2\pi n)\ n+O\Big(\ln^2 n\Big)\Big)\Big(\Big(\frac{1}{en}+\frac{1}{2en^2}\ln(2\pi n)\ +O\Big(\frac{\ln^2 n}{n^3}\Big)\Big)\Big(\alpha+O\Big(\frac{1}{na^{2n}}\Big)\Big)\\
&\quad-\Big(\frac{1}{e(n+1)}+\frac{1}{2e(n+1)^2}\ln(2\pi(n+1))\ +O\Big(\frac{\ln^2 n}{n^3}\Big)\Big)\Big(\alpha-\frac{\alpha\ln 5}{2n}+O\Big(\frac{1}{n^2}\Big)\Big)\Big)\\
&=\Big(\frac{1}{e^2}n^2+O\Big(n\ln n\Big)\Big)\Big(\frac{\alpha}{en(n+1)}+\frac{\alpha\ln 5}{2en(n+1)}+O\Big(\frac{\ln^2 n}{n^3}\Big)\Big)\\
&\quad\to\alpha\frac{1+\ln\sqrt{5}}{e^3},
\end{aligned}
$$

as n tends to infinity.

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Ángel Plaza, Raphael Schumacher, David Terr, Andrés Ventas, and the proposers.

<u>H-899</u> Proposed by Robert Frontczak, Stuttgart, Germany
(Vol. 60, No. 2, May 2022)

Show that

$$\sum_{n=1}^{\infty}\sinh^{-1}\left(\frac{1}{5F_nF_{n+1}}(L_{n+1}\sqrt{2L_{2n}}-L_n\sqrt{2L_{2n+2}})\right)=\frac{1}{2}\ln\left(\frac{(3+2\sqrt{2})(7-2\sqrt{6})}{5}\right).$$

Solution by Brian Bradie, Newport News, VA

Using the identity $L_{2n}=\frac{1}{2}(5F_n^2+L_n^2)$, it follows that

$$
\begin{aligned}
&\frac{1}{5F_nF_{n+1}}(L_{n+1}\sqrt{2L_{2n}}-L_n\sqrt{2L_{2n+2}})\\
&\qquad=\frac{L_{n+1}}{\sqrt{5}F_{n+1}}\cdot\sqrt{\frac{5F_n^2+L_n^2}{5F_n^2}}-\frac{L_n}{\sqrt{5}F_n}\cdot\sqrt{\frac{5F_{n+1}^2+L_{n+1}^2}{5F_{n+1}^2}}\\
&\qquad=\frac{L_{n+1}}{\sqrt{5}F_{n+1}}\cdot\sqrt{1+\left(\frac{L_n}{\sqrt{5}F_n}\right)^2}-\frac{L_n}{\sqrt{5}F_n}\cdot\sqrt{1+\left(\frac{L_{n+1}}{\sqrt{5}F_{n+1}}\right)^2}.
\end{aligned}
$$

Therefore,

$$\sinh^{-1}\left(\frac{1}{5F_nF_{n+1}}(L_{n+1}\sqrt{2L_{2n}}-L_n\sqrt{2L_{2n+2}})\right)=\sinh^{-1}\left(\frac{L_{n+1}}{\sqrt{5}F_{n+1}}\right)-\sinh^{-1}\left(\frac{L_n}{\sqrt{5}F_n}\right)$$

and the desired sum telescopes. In particular,

$$\begin{aligned}\sum_{n=1}^{\infty}\sinh^{-1}&\left(\frac{1}{5F_nF_{n+1}}(L_{n+1}\sqrt{2L_{2n}}-L_n\sqrt{2L_{2n+2}})\right)\\ &= \lim_{n\to\infty}\sinh^{-1}\left(\frac{L_{n+1}}{\sqrt{5}F_{n+1}}\right)-\sinh^{-1}\left(\frac{1}{\sqrt{5}}\right)\\ &= \sinh^{-1}(1)-\sinh^{-1}\left(\frac{1}{\sqrt{5}}\right)=\ln(1+\sqrt{2})-\ln\left(\frac{1+\sqrt{6}}{\sqrt{5}}\right)\\ &= \ln\left(\frac{\sqrt{5}(1+\sqrt{2})}{1+\sqrt{6}}\right)=\ln\left(\frac{(1+\sqrt{2})(\sqrt{6}-1)}{\sqrt{5}}\right)\\ &= \frac{1}{2}\ln\left(\frac{(1+\sqrt{2})^2(\sqrt{6}-1)^2}{5}\right)=\frac{1}{2}\ln\left(\frac{(3+2\sqrt{2})(7-2\sqrt{6})}{5}\right).\end{aligned}$$

Also solved by Dmitry Fleischman, Ángel Plaza, Albert Stadler, Séan M. Stewart, David Terr, and the proposer.

H-900 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 60, No. 2, May 2022)

Let $\mathbf{i}=\sqrt{-1}$. For any odd integer $m\ge 1$, prove that

$$\sum_{n=0}^{\infty}\frac{1}{L_{m(2n+1)}+L_{2m}\mathbf{i}}=\frac{2}{5F_mF_{2m}}-\frac{\mathbf{i}}{\sqrt{5}F_{2m}}.$$

Solution by Ángel Plaza, Gran Canaria, Spain

Because the proposed series is absolutely convergent and

$$\frac{1}{L_{m(2n+1)}+L_{2m}\mathbf{i}}=\frac{L_{m(2n+1)}-L_{2m}\mathbf{i}}{L_{m(2n+1)}^2+L_{2m}^2},$$

it is enough to prove that

$$\sum_{n=0}^{\infty}\frac{L_{m(2n+1)}}{L_{m(2n+1)}^2+L_{2m}^2}=\frac{2}{5F_mF_{2m}},\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{L_{2m}}{L_{m(2n+1)}^2+L_{2m}^2}=\frac{1}{\sqrt{5}F_{2m}}.$$

Because $m\ge 1$ is odd, $L_{m(2n+1)}^2=L_{2m(2n+1)}-2$, and $L_{2m}^2=L_{4m}+2$, and the expressions to prove become

$$\sum_{n=0}^{\infty}\frac{L_{m(2n+1)}}{L_{2m(2n+1)}+L_{4m}}=\frac{2}{5F_mF_{2m}},\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{L_{2m}}{L_{2m(2n+1)}+L_{4m}}=\frac{1}{\sqrt{5}F_{2m}}.$$

Note that $F_mF_{2m}=\dfrac{L_{3m}+L_m}{5}$, so the first sum to be proved may be written as

$$\sum_{n=0}^{\infty}\frac{L_{m(2n+1)}(L_{3m}+L_m)}{L_{2m(2n+1)}+L_{4m}}=2,$$

and because m is odd, $L_{m(2n+1)}(L_{3m}+L_m) = L_{2m(n+2)} - L_{2m(n-1)} + L_{2m(n+1)} - L_{2mn}$. If we rename $\alpha^{2m} = a$, and $\beta^{2m} = b$, the first sum becomes

$$\sum_{n=0}^{\infty} \frac{a^{n+2}+b^{n+2}+a^{n+1}+b^{n+1}-a^{n-1}-b^{n-1}-a^n-b^n}{a^{2n+1}+b^{2n+1}+a^2+b^2} = 2.$$

Note that

$$\begin{aligned}
&\frac{a^{n+2}+b^{n+2}+a^{n+1}+b^{n+1}-a^{n-1}-b^{n-1}-a^n-b^n}{a^{2n+1}+b^{2n+1}+a^2+b^2} \\
&= \frac{a^{3n+3}+a^{n-1}+a^{3n+2}+a^n-a^{3n}-a^{n+2}-a^{3n+1}-a^{n+1}}{a^{4n+2}+a^{2n+3}+a^{2n-1}+1} \\
&= (a+1)\left(\frac{a^n}{a^{2n}+a} - \frac{a^{n+1}}{a^{2n+3}+1}\right).
\end{aligned}$$

Therefore, the sum equals

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{a^{n+2}+b^{n+2}+a^{n+1}+b^{n+1}-a^{n-1}-b^{n-1}-a^n-b^n}{a^{2n+1}+b^{2n+1}+a^2+b^2} \\
&\qquad = (a+1)\sum_{n=0}^{\infty}\left(\frac{a^n}{a^{2n}+a} - \frac{a^{n+1}}{a^{2n+3}+1}\right) \\
&\qquad = (a+1)\left(\frac{1}{a+1} + \frac{a}{a^2+a}\right) = 2.
\end{aligned}$$

Analogously, the second sum may be written as

$$\sum_{n=0}^{\infty} \frac{a+b}{a^{2n+1}+b^{2n+1}+a^2+b^2} = \frac{1}{a-b},$$

or equivalently,

$$\sum_{n=0}^{\infty} \frac{a^2-b^2}{a^{2n+1}+b^{2n+1}+a^2+b^2} = 1.$$

This is true because

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a^2-b^2}{a^{2n+1}+b^{2n+1}+a^2+b^2} &= \sum_{n=0}^{\infty} \frac{a^{2n+3}-a^{2n-1}}{a^{4n+2}+a^{2n+3}+a^{2n-1}+1} \\
&= \sum_{n=0}^{\infty}\left(\frac{a}{a^{2n}+a} - \frac{1}{a^{2n+3}+1}\right) \text{ (which telescopes)} \\
&= \frac{a}{1+a} + \frac{a}{a^2+a} = 1.
\end{aligned}$$

Also solved by Brian Bradie, Dmitry Fleischman, Won Kyun Jeong, Albert Stadler, and the proposer.