ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-708 **Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.**

Let n be a positive integer. Prove that

$$\left(\frac{n}{F_n^2+F_{n+1}^2}\right)^2+\left(\frac{1}{4n^2}\prod_{k=1}^{n}\frac{1}{F_k^4}\right)\left(\sum_{k=1}^{n}(F_k^4-1)^{1/2}\right)^2\le\frac{1}{4}.$$

H-709 **Proposed by Ovidiu Furdui, Campia Turzii, Romania**

a) Let a be a positive real number. Calculate,

$$\lim_{n\to\infty}a^n\left(n-\zeta(2)-\zeta(3)-\cdots-\zeta(n)\right),$$

where ζ is the Riemann zeta function.

b) Let a be a real number such that $|a|<2$. Prove that,

$$\sum_{n=2}^{\infty}a^n\left(n-\zeta(2)-\zeta(3)-\cdots-\zeta(n)\right)=a\left(\frac{\Psi(2-a)+\gamma}{1-a}-1\right),$$

where Ψ denotes the Digamma function.

H-710 **Proposed by Emeric Deutsch, Polytechnic Institute of NYU, Brooklyn, NY**

Let $a_{n,k}$ denote the number of ternary words (i.e., finite sequences of 0's, 1's and 2's) of length n and having k occurrences of 01's. Find the generating function $G(t,z)=\sum_{k\ge0,n\ge0}a_{n,k}t^kz^n$.

H-711 **Proposed by Hideyuki Ohtsuka, Saitama, Japan**

Let $R_n=\sqrt{F_{n+3}}+\sqrt{F_{n+4}}$ for $n\ge0$. Prove that

$$R_n-R_2+2\le\sum_{k=1}^{n}\sqrt{F_k}\le R_n-R_1+1.$$

SOLUTIONS

Series with Powers of the Golden Section

H-687 Proposed by G. C. Greubel, Newport News, VA (Vol. 47, No. 2, May 2009/2010)

i) Show that

$$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}-\frac{\beta^2}{5n+3}-\frac{\beta^4}{5n+4}-\frac{\beta^5}{5n+5}\right](-\beta^5)^n=\pi\left(\frac{\alpha^6}{5^5}\right)^{\frac{1}{4}}.$$

ii) From the series in i) and H-669 (corrected) show that

ii.1) $$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}+\frac{1}{5n+2}-\frac{\beta^2}{5n+4}-\frac{\beta^4}{5n+5}\right](-\beta^5)^n=\pi\left(\frac{\alpha^2}{5}\right)^{\frac{5}{4}};$$

ii.2) $$\sum_{n=0}^{\infty}\left[\frac{\alpha^3}{5n+2}+\frac{\alpha}{5n+3}-\frac{1}{5n+4}-\frac{1}{5n+5}\right](-\beta^5)^n=\pi\left(\frac{\alpha^{14}}{5^5}\right)^{\frac{1}{4}};$$

ii.3) $$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}+\frac{\beta^2}{5n+2}+\frac{\beta^3}{5n+3}+\frac{\beta^3}{5n+4}\right](-\beta^5)^n=2\pi\left(\frac{\alpha^2}{5^5}\right)^{\frac{1}{4}}.$$

Solution by the proposer

Part I. Consider the series

$$S(\theta)=\sum_{n=1}^{\infty}\frac{\sin(4n-3)\theta}{n}(2\cos\theta)^n. \tag{1}$$

The sine term can be expanded to provide

$$S(\theta)=\cos 3\theta\sum_{n=1}^{\infty}\frac{\sin 4n\theta}{n}(2\cos\theta)^n-\sin 3\theta\sum_{n=1}^{\infty}\frac{\cos 4n\theta}{n}(2\cos\theta)^n. \tag{2}$$

By using the two series

$$\sum_{n=1}^{\infty}\frac{r^n}{n}\cos n\theta=-\frac{1}{2}\ln\left(1-2r\cos\theta+r^2\right)\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{r^n}{n}\sin n\theta=\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right),$$

equation (2) becomes

$$S(\theta)=\frac{1}{2}\sin 3\theta\ln\left(1-4\cos\theta\cos 4\theta+4\cos^2\theta\right)+\cos 3\theta\tan^{-1}\left(\frac{2\cos\theta\sin 4\theta}{1-2\cos\theta\cos 4\theta}\right).$$

By setting $\theta=2\pi/5$ we may obtain that $S(2\pi/5)$ equals

$$\frac{1}{2}\sin\frac{6\pi}{5}\ln\left(1-4\cos\frac{2\pi}{5}\cos\frac{8\pi}{5}+4\cos^2\frac{2\pi}{5}\right)+\cos\frac{6\pi}{5}\tan^{-1}\left(\frac{2\cos\frac{2\pi}{5}\sin\frac{8\pi}{5}}{1-2\cos\frac{2\pi}{5}\cos\frac{8\pi}{5}}\right)$$

$$=-\sin\frac{\pi}{5}\ln\left(1+4\cos\frac{2\pi}{5}\cos\frac{3\pi}{5}+4\cos^2\frac{2\pi}{5}\right)-\cos\frac{\pi}{5}\tan^{-1}\left(\frac{-2\cos\frac{2\pi}{5}\sin\frac{3\pi}{5}}{1+2\cos\frac{2\pi}{5}\cos\frac{3\pi}{5}}\right). \tag{3}$$

The values of the sine and cosine functions are given by

$$\sin\frac{\pi}{5}=\frac{\sqrt{-\beta\sqrt{5}}}{2},\qquad \cos\frac{\pi}{5}=\frac{\alpha}{2},\qquad \sin\frac{2\pi}{5}=\frac{\sqrt{\alpha\sqrt{5}}}{2},\qquad \cos\frac{2\pi}{5}=-\frac{\beta}{2},$$
$$\sin\frac{3\pi}{5}=\frac{\sqrt{\alpha\sqrt{5}}}{2},\qquad \cos\frac{3\pi}{5}=\frac{\beta}{2},\qquad \sin\frac{4\pi}{5}=\frac{\sqrt{-\beta\sqrt{5}}}{2}.$$

Using the above values for the sine and cosine functions in (3), we get

$$\begin{aligned}S\left(\frac{2\pi}{5}\right)&=-\frac{1}{2}\sin\frac{\pi}{5}\ln(1-\beta^2+\beta^2)-\frac{\alpha}{2}\tan^{-1}\left(\frac{2\beta}{1-\beta^2}\sin\frac{3\pi}{5}\right)\\&=-\frac{\alpha}{2}\tan^{-1}\left(\frac{2\beta}{\alpha}\sin\frac{3\pi}{5}\right)=-\frac{\alpha}{2}\tan^{-1}\left(\beta\sqrt{\frac{\sqrt{5}}{\alpha}}\right).\end{aligned}\tag{4}$$

Now, it can be seen that

$$\beta\sqrt{\frac{\sqrt{5}}{\alpha}}=-\sqrt{\frac{\sqrt{5}}{\alpha^3}}=-\sqrt{\frac{-\beta\sqrt{5}}{\alpha^2}}=-\frac{\frac{1}{2}\sqrt{-\beta\sqrt{5}}}{\frac{1}{2}\alpha}=-\tan\frac{\pi}{5}.$$

Using the above value into (4), we have

$$S\left(\frac{2\pi}{5}\right)=-\frac{\alpha}{2}\tan^{-1}\left(-\tan\frac{\pi}{5}\right)=\frac{\pi\alpha}{10}.\tag{5}$$

Alternatively, from equation (1), we have

$$S\left(\frac{2\pi}{5}\right)=\sum_{n=1}^{\infty}\frac{(-\beta)^n}{n}\sin\frac{2(4n-3)\pi}{5}.$$

By writing out the terms of the series we have the following:

$$\begin{aligned}S\left(\frac{2\pi}{5}\right)&=\left[(-\beta)\sin\frac{2\pi}{5}+\frac{(-\beta)^3}{3}\sin\frac{18\pi}{5}+\frac{(-\beta)^4}{4}\sin\frac{26\pi}{5}+\frac{(-\beta)^5}{5}\sin\frac{34\pi}{5}\right]\\&\quad+\left[\frac{(-\beta)^6}{6}\sin\frac{42\pi}{5}+\cdots\right]+\cdots\\&=\left[(-\beta)\sin\frac{2\pi}{5}-\frac{(-\beta)^3}{3}\sin\frac{3\pi}{5}-\frac{(-\beta)^4}{4}\sin\frac{\pi}{5}+\frac{(-\beta)^5}{5}\sin\frac{4\pi}{5}\right]\\&\quad+\left[\frac{(-\beta)^6}{6}\sin\frac{2\pi}{5}+\cdots\right]+\cdots\\&=\sum_{n=0}^{\infty}\left[\frac{(-\beta)^{5n+1}}{5n+1}\sin\frac{2\pi}{5}-\frac{(-\beta)^{5n+3}}{5n+3}\sin\frac{3\pi}{5}-\frac{(-\beta)^{5n+4}}{5n+4}\sin\frac{\pi}{5}+\frac{(-\beta)^{5n+5}}{5n+5}\sin\frac{4\pi}{5}\right]\\&=\sum_{n=0}^{\infty}\left[\frac{-\beta\sin\frac{2\pi}{5}}{5n+1}+\frac{\beta^3\sin\frac{3\pi}{5}}{5n+3}-\frac{\beta^4\sin\frac{\pi}{5}}{5n+4}-\frac{\beta^5\sin\frac{4\pi}{5}}{5n+5}\right]\left(-\beta^5\right)^n\\&=\sin\frac{\pi}{5}\cdot\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}-\frac{\beta^2}{5n+3}-\frac{\beta^4}{5n+4}-\frac{\beta^5}{5n+5}\right]\left(-\beta^5\right)^n.\end{aligned}\tag{6}$$

Equating the results for $S\left(\frac{2\pi}{5}\right)$ from equations (5) and (6) provides

$$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}-\frac{\beta^2}{5n+3}-\frac{\beta^4}{5n+4}-\frac{\beta^5}{5n+5}\right]\left(-\beta^5\right)^n=\pi\left(\frac{\alpha^6}{5^5}\right)^{\frac{1}{4}}, \tag{7}$$

which is the desired result.

Part II. The value of the series in question is obtained from the difference of the two series. The values of the series are given by

$$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}+\frac{2}{5n+2}+\frac{\beta^2}{5n+3}+\frac{\beta}{5n+4}-\frac{\beta^2}{5n+5}\right]\left(-\beta^5\right)^n=\pi\left(\frac{\alpha^6}{5^3}\right)^{\frac{1}{4}} \tag{8}$$

from problem H-669(corrected), and

$$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}-\frac{\beta^2}{5n+3}-\frac{\beta^4}{5n+4}-\frac{\beta^5}{5n+5}\right]\left(-\beta^5\right)^n=\pi\left(\frac{\alpha^6}{5^5}\right)^{\frac{1}{4}} \tag{9}$$

from (7) above.

Now adding equations (8) and (9) yields

$$\begin{aligned}
S_1&=\sum_{n=0}^{\infty}\left[\frac{2}{5n+1}+\frac{2}{5n+2}+\frac{\beta(1-\beta^3)}{5n+4}-\frac{\beta^2(1+\beta^3)}{5n+5}\right]\left(-\beta^5\right)^n\\
&=\sum_{n=0}^{\infty}\left[\frac{2}{5n+1}+\frac{2}{5n+2}-\frac{2\beta^2}{5n+4}-\frac{2\beta^4}{5n+5}\right]\left(-\beta^5\right)^n\\
&=2\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}+\frac{1}{5n+2}-\frac{\beta^2}{5n+4}-\frac{\beta^4}{5n+5}\right]\left(-\beta^5\right)^n. \qquad (10)
\end{aligned}$$

The value of the series (10) is given by

$$S_1=\pi\left(\frac{\alpha^6}{5^3}\right)^{\frac{1}{4}}+\pi\left(\frac{\alpha^6}{5^5}\right)^{\frac{1}{4}}=\pi\left(\frac{\alpha^6}{5^5}\right)^{\frac{1}{4}}(1+\sqrt{5})=2\pi\left(\frac{\alpha^{10}}{5^5}\right)^{\frac{1}{4}}=2\pi\left(\frac{\alpha^2}{5}\right)^{\frac{5}{4}}. \tag{11}$$

From (10) and (11), we have

$$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}+\frac{1}{5n+2}-\frac{\beta^2}{5n+4}-\frac{\beta^4}{5n+5}\right]\left(-\beta^5\right)^n=\pi\left(\frac{\alpha^2}{5}\right)^{\frac{5}{4}},$$

which is the desired result.

The second series can be obtained from the subtraction of (9) from (8). For this, we have

$$\begin{aligned}
S_2&=\sum_{n=0}^{\infty}\left[\frac{2}{5n+2}+\frac{2\beta^2}{5n+3}+\frac{\beta(1+\beta^3)}{5n+4}-\frac{\beta^2(1-\beta^3)}{5n+5}\right]\left(-\beta^5\right)^n\\
&=\sum_{n=0}^{\infty}\left[\frac{2}{5n+2}+\frac{2\beta^2}{5n+3}+\frac{2\beta^3}{5n+4}+\frac{2\beta^3}{5n+5}\right]\left(-\beta^5\right)^n\\
&=-2\beta^3\sum_{n=0}^{\infty}\left[\frac{\alpha^3}{5n+2}+\frac{\alpha}{5n+3}-\frac{1}{5n+4}-\frac{1}{5n+5}\right]\left(-\beta^5\right)^n. \qquad (12)
\end{aligned}$$

The value is given by

$$S_2 = \pi \left(\frac{\alpha^6}{5^5}\right)^{\frac{1}{4}} (\sqrt{5} - 1) = -2\beta^3 \pi \left(\frac{\alpha^{14}}{5^5}\right)^{\frac{1}{4}}. \tag{13}$$

Equating the equations (12) and (13) for S_2 yields

$$\sum_{n=0}^{\infty} \left[\frac{\alpha^3}{5n+2} + \frac{\alpha}{5n+3} - \frac{1}{5n+4} - \frac{1}{5n+5}\right] (-\beta^5)^n = \pi \left(\frac{\alpha^{14}}{5^5}\right)^{\frac{1}{4}}, \tag{14}$$

which is the desired result. The third series can be obtained by multiplying equation (14) by $\frac{\beta}{\alpha}$ and adding the result to equation (12). When this is done, the resulting series is given by

$$\sum_{n=0}^{\infty} \left[\frac{1}{5n+1} + \frac{\beta^2}{5n+2} + \frac{\beta^3}{5n+3} + \frac{\beta^3}{5n+4}\right] (-\beta^5)^n = 2\pi \left(\frac{\alpha^2}{5^5}\right)^{\frac{1}{4}}.$$

Also solved by Paul S. Bruckman and Kenneth B. Davenport.

Congruence with Fibonacci Numbers

H-689 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 47, No. 3, August 2009/2010)

For positive integers l, m, and n such that $l \neq m$, prove that

$$F_m^n F_{ln} \equiv F_l^n F_{mn} \pmod{F_{m-l}}.$$

Solution by the proposer

$$\sum_{k=0}^{n} \binom{n}{k} F_l^k ((-1)^l F_{m-l})^{n-k} \alpha^{mk} = \sum_{k=0}^{n} \binom{n}{k} (F_l \alpha^m)^k ((-1)^l F_{m-l})^{n-k}$$
$$= (F_l \alpha^m + (-1)^l F_{m-l})^n = (F_l(\alpha F_m + F_{m-1}) + F_{l+1} F_m - F_l F_{m+1})^n$$
$$= (\alpha F_l F_m + F_l F_{m-1} + F_{l+1} F_m - F_l F_{m+1})^n = (\alpha F_l F_m - F_l F_m + F_{l+1} F_m)^n$$
$$= F_m^n (\alpha F_l + F_{l-1})^n = F_m^n \alpha^{ln}.$$

Similarly,

$$\sum_{k=0}^{n} \binom{n}{k} F_l^k ((-1)^l F_{m-l})^{n-k} \beta^{mk} = F_m^n \beta^{ln}.$$

Therefore,

$$\sum_{k=0}^{n} \binom{n}{k} F_l^k ((-1)^l F_{m-l})^{n-k} F_{mk} = F_m^n F_{ln}.$$

Here, we have

$$F_m^n F_{ln} - F_l^n F_{mn} = \sum_{k=0}^{n-1} \binom{n}{k} F_l^k ((-1)^l F_{m-l})^{n-k} F_{mk} \equiv 0 \pmod{F_{m-l}}.$$

Thus,

$$F_m^n F_{ln} \equiv F_l^n F_{mn} \pmod{F_{m-l}}.$$

Also solved by Paul S. Bruckman.

Recurrences for Alternating Sums with Even Powers of the Fibonacci Numbers

H-690 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 47.3, August 2009/2010)

Let m and n be positive integers. Put

$$S_m(n) = \sum_{k=1}^{n} (-1)^{k(m+1)} F_k^{2m}.$$

Prove that

$$L_m S_m(n) = (-1)^{n(m+1)} S_1^m(n) - \sum_{i=1}^{\lfloor m/2 \rfloor} \sum_{r=1}^{i} \frac{m}{i} \binom{m-i-1}{i-1} \binom{i}{r} S_{m-r}(n).$$

Solution by the proposer

Let k and m be positive integers. The following identity is known (see [1, p. 63]). For real numbers a and b,

$$a^m + b^m = (a+b)^m + \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{m}{i} \binom{m-i-1}{i-1} (a+b)^{m-2i} (-ab)^i. \tag{15}$$

Setting $a = F_{k+1}$ and $b = -F_{k-1}$ in (15), we get

$$\begin{aligned}
F_{k+1}^m + (-F_{k-1})^m &= F_k^m + \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{m}{i} \binom{m-i-1}{i-1} F_k^{m-2i} (F_k^2 + (-1)^k)^i \\
&= F_k^m + \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{m}{i} \binom{m-i-1}{i-1} F_k^{m-2i} \sum_{r=0}^{i} \binom{i}{r} (-1)^{kr} F_k^{2(i-r)} \\
&= F_k^m \left\{ 1 + \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{m}{i} \binom{m-i-1}{i-1} \right\} \\
&\quad + \sum_{i=1}^{\lfloor m/2 \rfloor} \sum_{r=1}^{i} \frac{m}{i} \binom{m-i-1}{i-1} \binom{i}{r} (-1)^{kr} F_k^{m-2r}.
\end{aligned}$$

Setting $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$ in (15), we get

$$L_m = 1 + \sum_{i=1}^{\lfloor m/2 \rfloor} \frac{m}{i} \binom{m-i-1}{i-1}.$$

Therefore,

$$F_{k+1}^m + (-F_{k-1})^m = L_m F_k^m + \sum_{i=1}^{\lfloor m/2 \rfloor} \sum_{r=1}^{i} \frac{m}{i} \binom{m-i-1}{i-1} \binom{i}{r} (-1)^{kr} F_k^{m-2r}. \tag{16}$$

We have

$$
\begin{aligned}
(-1)^{n(m+1)}S_1^m(n) = (-1)^{n(m+1)}F_n^m F_{n+1}^m &= \sum_{k=1}^{n}\left((-1)^{k(m+1)}F_k^m F_{k+1}^m - (-1)^{(k-1)(m+1)}F_{k-1}^m F_k^m\right) \\
&= \sum_{k=1}^{n}(-1)^{k(m+1)}F_k^m\left(F_{k+1}^m + (-F_{k-1})^m\right) \\
&= \sum_{k=1}^{n}(-1)^{k(m+1)}F_k^m\left\{L_m F_k^m + \sum_{i=1}^{\lfloor m/2\rfloor}\sum_{r=1}^{i}\frac{m}{i}\binom{m-i-1}{i-1}\binom{i}{r}(-1)^{kr}F_k^{m-2r}\right\} \\
&= (-1)^{k(m+1)}\sum_{k=1}^{n}\left\{F_k^{2m}L_m + \sum_{i=1}^{\lfloor m/2\rfloor}\sum_{r=1}^{i}\frac{m}{i}\binom{m-i-1}{i-1}\binom{i}{r}(-1)^{kr}F_k^{2(m-r)}\right\} \\
&= L_m S_m(n) + \sum_{i=1}^{\lfloor m/2\rfloor}\sum_{r=1}^{i}\frac{m}{i}\binom{m-i-1}{i-1}\binom{i}{r}S_{m-r}(n).
\end{aligned}
$$

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

Also solved by Paul S. Bruckman.