ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-627 Proposed by Slavko Simic, Belgrade, Yugoslavia

Find all sequences $c = \{c_i\}_{i=1}^n$, $c_i = c_i(n)$ such that the inequality

$$|x^* - \sum_{i=1}^n c_i x_i| \leq \sqrt{n-1}\sqrt{\sum_{i=1}^n c_i x_i^2 - \big(\sum_{i=1}^n c_i x_i\big)^2},$$

holds for all sequences $x = \{x_i\}_{i=1}^n$ of arbitrary real numbers and arbitrary $x^* \in x$.

H-628 Proposed by Juan Pla, Paris, France

Let us consider the set S of all the sequences $\{U_n\}_{n\geq 0}$ satisfying a second order linear recurrence

$$U_{n+2} - aU_{n+1} + bU_n = 0,$$

with both a and b rational integers, and having only integral values. Prove that for infinitely many of these sequences their general term U_n is a sum of three cubes of integers for any value of the subscript n.

H-629 Proposed by Ernst Herrmann, Siegburg, Germany

Consider the sequence $(a_n)_{n\geq 0}$ of non-negative integers which are defined by $a_0 = a_1 = 0$, $a_2 = 1$ and by the recurrence relation $a_n = a_{n-2} + a_{n-3}$ if $n \geq 3$. Prove that the numbers of the sequence $(a_n)_{n\geq 0}$ can also be defined by the relation

$$-0.5 < a_{n+2} - a_{n+1}^2/a_n < 0.5$$

for all sufficiently large n; i.e., for all $n \geq n_0$. Thus, a_{n+2} is uniquely defined if a_n, a_{n+1} and a_{n+2} fulfill the relation. Determine the smallest possible value n_0.

H-630 Proposed by Mario Catalani, Torino, Italy

Let $F_n(x,y)$ be the bivariate Fibonacci polynomials, defined, for $n \geq 2$, by $F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y)$, where $F_0(x,y) = 0,\ F_1(x,y) = 1$. Assume $xy \neq 0$ and $x^2 + 4y \neq 0$.

1. Prove the following identity

$$x \sum_{k=0}^{n-1} \binom{2n-1-k}{k} (x^2+4y)^{n-k-1}(-y)^k = F_{2n}(x,y).$$

2. Let

$$a_n = \sum_{k=0}^{n-1} \binom{2n-1-k}{k} (-3)^{n-k-1}.$$

Find a recurrence and a closed form for a_n.

SOLUTIONS

Binomial sums yielding Fibonacci and Lucas numbers

H-570 Proposed by H.-J. Sieffert, Berlin, Germany
(Vol. 39, no. 1, February 2001)

Show that, for all positive integers n:

(a)

$$5^{n-1} F_{2n-1} = \sum_{\substack{k=0 \\ 5 \nmid 2n-k-1}}^{2n-1} (-1)^k \binom{4n-2}{k};$$

(b)

$$5^{n-1} L_{2n} = \sum_{\substack{k=0 \\ 5 \nmid 2n-k}}^{2n} (-1)^{k+1} \binom{4n}{k}.$$

Two closely related identities were given in H-518.

Solution by the proposer

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$ and $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$ for $n \geq 1$. If $A_n := i^{n-1}F_n(i\alpha)$, with $n \geq 0$, where $i = \sqrt{-1}$, then $A_{n+1} = -\alpha A_n - A_{n-1}$ and a simple induction argument yields

$$A_n = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 5, \\ 1 & \text{if } n \equiv 1 \pmod 5, \\ -\alpha & \text{if } n \equiv 2 \pmod 5, \\ \alpha & \text{if } n \equiv 3 \pmod 5, \\ -1 & \text{if } n \equiv 4 \pmod 5. \end{cases} \tag{1}$$

Let n be a positive integer. From H-518 (identity (7)), we know that, for all complex numbers x,

$$\sum_{k=0}^{n} \binom{2n}{n-k} F_k(x)^2 = (x^2+4)^{n-1}.$$

Since $4 - \alpha^2 = -\sqrt{5}\beta$, with $x = i\alpha$, we find

$$\sum_{k=0}^{n} (-1)^{k-1} \binom{2n}{n-k} A_k^2 = \left(-\sqrt{5}\beta\right)^{n-1}.$$

Hence, by (1),

$$\sum_{k=0}^{n} (-1)^{k-1} \binom{2n}{n-k} c_k + \alpha^2 \sum_{k=0}^{n} (-1)^{k-1} \binom{2n}{n-k} d_k = \left(-\sqrt{5}\beta\right)^{n-1}, \tag{2}$$

where

$$c_k = \begin{cases} 1 & \text{if } k \equiv 1 \text{ or } 4 \pmod 5, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad d_k = \begin{cases} 1 & \text{if } k \equiv 2 \text{ or } 3 \pmod 5, \\ 0 & \text{otherwise,} \end{cases}$$

Let S_n and T_n denote the first and the second sum on the left hand side of (2), respectively. Then, $S_n + \alpha^2 T_n = (-\sqrt{5}\beta)^{n-1}$. Since $\alpha^2 = (3+\sqrt{5})/2$, $\beta^{n-1} = (L_{n-1} - \sqrt{5}F_{n-1})/2$, and since $\sqrt{5}$ is an irrational number, from (2), we find

$$2S_n + 3T_n = \begin{cases} 5^{(n-1)/2} L_{n-1} & \text{if } n \text{ is odd,} \\ 5^{n/2} F_{n-1} & \text{if } n \text{ is even,} \end{cases} \tag{3}$$

and

$$T_n = \begin{cases} -5^{(n-1)/2} F_{n-1} & \text{if } n \text{ is odd,} \\ \\ -5^{(n-2)/2} L_{n-1} & \text{if } n \text{ is even.} \end{cases} \tag{4}$$

Substracting (4) from (3), noting that $L_{n-1} + F_{n-1} = 2F_n$ and $5F_{n-1} + L_{n-1} = 2L_n$, and dividing by 2, we obtain

$$S_n + T_n = \begin{cases} 5^{(n-1)/2} F_n & \text{if } n \text{ is odd,} \\ \\ 5^{(n-2)/2} L_n & \text{if } n \text{ is even,} \end{cases}$$

The stated identities now follow if writing $2n-1$ respectively $2n$ instead of n; note that $c_k + d_k = 1$ if $5 \not| k$, and $c_k + d_k = 0$ if $5|k$. The known and easily verified identity

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}, \qquad 1 \le m \le n,$$

with $m = 2n-1$ and n replaced by $4n-2$, implies that (a) is equivalent to

$$5^{n-1} F_{2n-1} = -\binom{4n-3}{2n-1} + \sum_{j=0}^{\lfloor (2n-1)/5 \rfloor} (-1)^j \binom{4n-2}{2n-5j-1}.$$

Similarly, (b) can be rewritten as

$$5^{n-1} L_{2n} = -\binom{4n-1}{2n} + \sum_{j=0}^{\lfloor 2n/5 \rfloor} (-1)^j \binom{4n}{2n-5j}.$$

A Cyclic Determinant

H-615 Proposed by Paul S. Bruckman, Sointula, Canada
(Vol. 42, no. 4, November 2004)

Given $n \geq 1$ and complex numbers $x_0,\ x_1,\ \ldots,\ x_{n-1}$, define the "cyclical" matrix

$$\boldsymbol{A}_n = \begin{vmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \ldots & x_{n-3} & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \ldots & x_{n-4} & x_{n-3} \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_0 \end{vmatrix}.$$

Let D_n denote the determinant of $\boldsymbol{A}_n$, and $s_n = x_0 + x_1 + \cdots + x_{n-1}$. Prove that if the x_k's are integers such that $s_n \neq 0$, then $s_n | D_n$.

Solution by H.-J. Seiffert, Berlin, Germany

Adding the second, third, fourth, ... and last row to the first row in D_n, we arrive to the determinant E_n, whose entries of the first row are all equal to s_n, and having the same value as D_n. Extracting s_n gives $E_n = s_n F_n$, where F_n is obtained from E_n by replacing each entry of the first row by 1. If the x_k's are integers, then F_n is an integer, so that the desired result follows from $D_n = E_n = s_n F_n$.

Editor's Comment. Most solvers used the known fact that

$$D_n = \prod_{\{\zeta:\zeta^n=1\}} \Big(\sum_{i=0}^{n-1} x_i \zeta^i\Big),$$

where the above product is over all the roots ζ of order n of 1, together with the observation that s_n is the factor corresponding to $\zeta = 1$, to infer that if the x_k's are integers, then D_n/s_n is both a rational number and an algebraic integer, therefore an integer.

Also solved by Gökçen Alptekýn, Ovidiu Furdui, Russel J. Hendel and the proposer.

On the parity of the Catalan numbers

H-616 Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 42, no. 4, November 2004)

Let $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$, $n = 0,\ 1,\ \ldots$ be the nth Catalan number (known to be an integer).

Prove that C_n is odd if and only if $n = 2^u - 1$, where $u = 0,\ 1,\ \ldots$.

Editor's Comment. Several solvers pointed out that this is a known result. For example, H.-J. Seiffert refers to [1], while Bruce Sagan (via Emeric Deutsch) supplies five references including [1]. The problem was also solved by Art Benjamin, Charles Cook, Graham Lord and the proposer.

1. Ö. Eğecioğlu, "The parity of the Catalan numbers via lattice paths", The Fibonacci Quarterly **21.1** (1983): 65–66.

Correction. In the solution to problem H-606 (The Fibonacci Quarterly **43.1** (2005): 93–94) the correct formula for S_n on page 94 is

$$S_n = 2\sin\Big(\frac{(2n+1)}{6}\Big) - \frac{1}{2}(-1)^{\lfloor n/2 \rfloor}(1-(-1)^n).$$