ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2006. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1001 Proposed by Paul S. Bruckman, Canada

(a) Let $\theta(a,b) = \sin^{-1}\{(a^2-b^2)/(a^2+b^2)\}$ denote one of the acute angles of a Pythagorean triangle, where a and b are integers with $a > b > 0$. Given $\theta(a,b)$ and $\theta(c,d)$, show that there exists $\theta(e,f)$, where e and f are functions of a, b, c and d, such that

$$\theta(a,b) + \theta(c,d) + \theta(e,f) = \pi/2.$$

(b) As a special case, prove the following identity:

$$\sin^{-1}\{F_{n-1}F_{n+2}/F_{2n+1}\} + \sin^{-1}\{L_{n-1}L_{n+2}/5F_{2n+1}\} + \sin^{-1}\{3/5\} = \pi/2.$$

B-1002 **Proposed by H.-J. Seiffert, Berlin, Germany**

Prove the identity

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{F_{k+1}}{F_k}-\alpha\right)=\frac{1}{2}\left(-\beta+\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+1}}\right).$$

B-1003 **Proposed by Paul S. Bruckman, Canada**

Prove the identity

$$F_{n+2}L_{n+1}L_nF_{n-1}+L_{n+2}F_{n+1}F_nL_{n-1}=2(F_{2n+1})^2.$$

B-1004 **Proposed by José Luis Díaz-Barrero, Barcelona, Spain**

Let n be a positive integer. Calculate

$$\lim_{n\to\infty}\left\{\frac{1}{F_n}\left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{n}F_{|i+k|}}{\sum_{i=1}^{n}\sum_{k=1}^{n}F_{|i-k|}}\right)\right\}.$$

B-1005 **Proposed by H.-J. Seiffert, Berlin, Germany**

Prove that, for all integers k and n with $0\le k\le n$,

$$\sum_{j=0}^{2n-2k}(-1)^j\binom{2n+1}{j}\binom{2n-k-j}{k}F_j=0.$$

SOLUTIONS

A Closed Form for a Fibonacci Sum

B-986 Proposed by Br. J. Mahon, Australia (Vol. 42, no. 4, Nov. 2004)

Prove

$$\sum_{i=2}^{n} \frac{F_{2i-2}F_{2i}}{3\left(F_{2i}^2-1\right)\left(F_{2i+2}^2-1\right)} = \frac{-1}{8} + \frac{F_{2n}F_{2n+2}}{3\left(F_{2n+2}^2-1\right)}.$$

Solution by Steve Edwards, Southern Polytechnic State University, Marietta, GA

The formula is easily verified for small n, so we assume it to be true for values through n and proceed by induction:

$$\sum_{i=2}^{n+1} \frac{F_{2i-2}F_{2i}}{3(F_{2i}^2-1)(F_{2i+2}^2-1)} = \frac{-1}{8} + \frac{F_{2n}F_{2n+2}}{3(F_{2n+2}^2-1)} + \frac{F_{2n}F_{2n+2}}{3(F_{2n+2}^2-1)(F_{2n+4}^2-1)}$$

$$= \frac{-1}{8} + \frac{F_{2n}F_{2n+2}(F_{2n+4}^2-1)+F_{2n}F_{2n+2}}{3(F_{2n+2}^2-1)(F_{2n+4}^2-1)} = \frac{-1}{8} + \frac{F_{2n}F_{2n+2}F_{2n+4}^2}{3(F_{2n+2}^2-1)(F_{2n+4}^2-1)}.$$

But since $F_{2n}F_{2n+4} = (F_{2n+2}^2-1)$ (see e.g. [1]), this equals $\frac{-1}{8} + \frac{F_{2n+2}F_{2n+4}}{3(F_{2n+4}^2-1)}$.

Reference

1. Thomas Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, 2001.

All solutions were more or less similar to the featured one.

Also solved by Brian D. Beasley, Paul S. Bruckman, Kenneth B. Davenport, Ovidiu Furdui, George C. Greubel, Russell J. Hendel, Harris Kwong, Hradec Králové, H.-J. Seiffert, and the proposer.

Another Toss

B-987 Proposed by M.N. Deshpande and J.P. Shiwalkar, Nagpur, India (Vol. 42, no. 4, Nov. 2004)

An unbiased coin is tossed n times. Let A be the event that no two successive heads occur. Show $Pr(A) = (F_{n+2})/2^n$.

Solution by Maitland Rose and Michael Becker, University of South Carolina, Sumter

$Pr(A) =$(Number of sequences of coins for which no two heads are together)$/2^n$.

Fibonacci's tree diagram for his "rabbit model" applies directly to our present case. If A (adults) is replaced by T (tails) and B (babies) is replaced by H (heads) we may interpret no two successive heads as a baby does not produce a baby.

If we start with a "T", then for n tosses the number of outcomes with no two successive heads is F_{n+1}.

If we start with a "H", then for n tosses the number of outcomes with no two successive heads is F_n.

Therefore $Pr(A) = \frac{F_{n+1}+F_n}{2^n} = \frac{F_{n+2}}{2^n}$.

Comment. As most of the solvers pointed out, this problem is very well known. H.-J. Seiffert mentions that a proof was given recently in *The Mathematical Gazette*, 88, November 2004; by M. Griffiths. He also mentions that the problem is very similar to problem B-688 of *The Fibonacci Quarterly.* Pentti Haukkanen mentions that this problem occurs as an example or as an exercise, more or less in the same form, in most textbooks on combinatorics and discrete mathematics (see R. Grimaldi, example 10.12 or T. Koshy, Chapter 4 to name a few).

Also solved by Charles Ashbacher, Brian Beasley, Paul S. Bruckman, Steve Edwards, George C. Greubel, Pentti Haukkanen, Russell Hendel, Hradec Králové, Harris Kwong, Kathleen E. Lewis, Graham Lord, William Moser, H.-J. Seiffert, James Sellers, and the proposer.

It's All in the Parity

B-988 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA
(Vol. 42, no. 4, Nov. 2004)

Show that for an odd integer n and any integer k, $5(F_k^2 - F_{k-n}^2) + 4(-1)^k$ is the product of two Lucas numbers, while if n is an even integer, then $F_k^2 - F_{k-n}^2$ is the product of two Fibonacci numbers.

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

Using $\alpha\beta = -1$, we note the following:

$$\alpha^n\beta^{2k-n} = (\alpha\beta)^n\beta^{2k-2n} = (-1)^n\beta^{2k-2n};$$

$$\beta^n\alpha^{2k-n} = (\alpha\beta)^n\alpha^{2k-2n} = (-1)^n\alpha^{2k-2n}.$$

Then for even n, using the above results along with the observation that $(-1)^k = (-1)^{k-n}$, we obtain

$$F_k^2 - F_{k-n}^2 = \frac{1}{5}\left[\alpha^{2k} - 2(-1)^k + \beta^{2k} - \alpha^{2k-2n} + 2(-1)^{k-n} - \beta^{2k-2n}\right]$$

$$= \frac{1}{5}(\alpha^n - \beta^n)(\alpha^{2k-n} - \beta^{2k-n}) = F_nF_{2k-n}.$$

Similarly, for odd n, we obtain

$$F_k^2 - F_{k-n}^2 = \frac{1}{5}\left[\alpha^{2k} + \beta^{2k} - \alpha^{2k-2n} - \beta^{2k-2n} - 4(-1)^k\right]$$

and thus

$$5(F_k^2 - F_{k-n}^2) + 4(-1)^k = \alpha^{2k} + \beta^{2k} - \alpha^{2k-2n} - \beta^{2k-2n}$$

$$= (\alpha^n + \beta^n)(\alpha^{2k-n} + \beta^{2k-n}) = L_n L_{2k-n}.$$

Also solved by Paul S. Bruckman, Charles K. Cook, Ovidiu Furdui, George C. Greubel, Russell J. Hendel, Hradec Králové, Harris Kwong, William Moser, Maitland Rose, H.-J. Seiffert, James Sellers, and the proposer.

A Lower Bound for a Fibonacci Sum

B-989 Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Catalunya, Barcelona, Spain (Vol. 42, no. 4, Nov. 2004)

Let n be a positive integer. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{F_k C(n,k)}\right)^n \geq e^{n-F_{2n}}.$$

Solution by H.-J. Seiffert, Berlin, Germany

The Cauchy-Schwarz Inequality gives

$$\left(\sum_{k=1}^{n}\frac{1}{F_k C(n,k)}\right)\left(\sum_{k=1}^{n} F_k C(n,k)\right) \geq n^2.$$

Since (see, for example, P. Haukkanen. "On a Binomial Sum for the Fibonacci and Related Numbers." *The Fibonacci Quarterly* 34.4 (1996): 326-31, (1) and (2))

$$\sum_{k=1}^{n} F_k C(n,k) = F_{2n},$$

we obtain

$$\sum_{k=1}^{n}\frac{1}{F_k C(n,k)} \geq \frac{n^2}{F_{2n}}.$$

The known inequality $e^x \geq 1 + x,\ x \in R$, with $x = F_{2n}/n - 1$ implies

$$\frac{n}{F_{2n}} \geq e^{1-F_{2n}/n}.$$

The stated inequality follows.

Also solved by Paul S. Bruckman, Steve Edwards, Ovidiu Furdui, George C. Greubel, Russell J. Hendel, Hradéc Králové, Harris Kwong, and the proposer.

Two Binomial-Type Identities

B-990 Proposed by Mario Catalani, University of Torino, Torino, Italy (Vol. 42, no. 4, Nov. 2004)

Let $F_n(x,y)$ and $L_n(x,y)$ be the bivariate Fibonacci and Lucas polynomials, defined, respectively, by $F_n(x,y) = xF_{n-1}(x,y)+yF_{n-2}(x,y), F_0(x,y) = 0, F_1(x,y) = 1$ and $L_n(x,y) = xL_{n-1}(x,y)+yL_{n-2}(x,y), L_0(x,y) = 2, L_1(x,y) = x$. Assume $x^2+4y \neq 0$. Prove the following identities

A) $$\sum_{k=0}^{n} \binom{n}{k} F_k(x,y)F_{n-k}(x,y) = \frac{1}{x^2+4y}(2^n L_n(x,y) - 2x^n),$$

B) $$\sum_{k=0}^{n} \binom{n}{k} L_k(x,y)L_{n-k}(x,y) = 2^n L_n(x,y) + 2x^n.$$

Solution by Paul S. Bruckman, Sointula, BC, and Harris Kwong, SUNY Fredonia, Fredonia, NY (independently)

Let $\alpha = \alpha(x,y) = (x + \sqrt{x^2+4y})/2$ and $\beta = \beta(x,y) = (x - \sqrt{x^2+4y})/2$. We can derive the Binet's formulas for $F_n(x,y)$ and $L_n(x,y)$ in the same manner they were derived for the ordinary Fibonacci and Lucas numbers. We find

$$F_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{x^2+4y}} \quad \text{and} \quad L_n(x,y) = \alpha^n + \beta^n.$$

A) $$\begin{aligned}(x^2+4y)\sum_{k=0}^{n} \binom{n}{k} F_k(x,y)F_{n-k}(x,y) &= \sum_{k=0}^{n} \binom{n}{k} (\alpha^k - \beta^k)(\alpha^{n-k} - \beta^{n-k}) \\ &= \sum_{k=0}^{n} \binom{n}{k} [(\alpha^n + \beta^n) - (\alpha^k \beta^{n-k} + \alpha^{n-k}\beta^k)] \\ &= (\alpha^n + \beta^n)\sum_{k=0}^{n} \binom{n}{k} - 2(\alpha+\beta)^n \\ &= 2^n L_n(x,y) - 2x^n.\end{aligned}$$

B) In a similar manner, we also find

$$\sum_{k=0}^{n} \binom{n}{k} L_k(x,y)L_{n-k}(x,y) = 2^n L_n(x,y) + 2x^n.$$

Also solved by Kenneth B. Davenport, Ovidiu Furdui, George C. Greubel, Hradec Králové, H.-J. Seiffert, and the proposer.

We wish to belatedly acknowledge the solution to problem B-976 by Scott Brown.