PERIODIC PROPERTIES OF FIBONACCI SUMMATIONS

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INTRODUCTION

It is well known that if we take the terms of the Fibonacci sequence modulo m that the least positive residues form a periodic sequence. This paper will consider the summation of functions of such residues taken over a period with the further limitations that for most of the results the modulus considered is a prime and the total degree of the product being summed is less than the prime modulus.

NOTATION

We employ the usual notation F_i to signify the terms of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, ... The letter p represents a prime and m any positive integer.

We shall be considering summations such as:

$$\sum_P F_i^3 F_{i-3}^2 F_{i-5}^4$$

where the subscripts of the Fibonacci numbers in the product differ from each other by fixed integers; where the summation is taken over a period for a given modulus p, this being indicated by having P below the summation sign; and where the total degree n of the product being summed is the sum of the exponents of the Fibonacci numbers.

Theorem 1. The summation of the residues of the Fibonacci sequence over a period is congruent to zero modulo m.

Proof: From the basic relation for the Fibonacci sequence

$$F_i = F_{i-1} + F_{i-2}$$

it follows that

$$\sum_P F_i = \sum_P F_{i-1} + \sum_P F_{i-2}$$

From the nature of periodicity, it is clear that the summation over a period will always be congruent to the same quantity for a given modulus regardless of where we start in the sequence. Thus

$$\sum_P F_i \equiv \sum_P F_{i-1} \equiv \sum_P F_{i-2} \pmod{m}$$

so that

$$\sum_P F_i \equiv 2 \sum_P F_i \pmod{m}$$

which leads to the conclusion that

$$\sum_P F_i \equiv 0 \pmod{m} .$$

Theorem 2. The summations

$$\sum_P F_i^2 \quad \text{and} \quad \sum_P F_i F_{i-1}$$

are congruent to zero modulo any prime with the possible exception of 2.

Proof. For convenience we shall replace

$$\sum_P F_i^2 \quad \text{by} \quad a \quad \text{and} \quad \sum_P F_i F_{i-1} \quad \text{by} \quad b$$

noting once more that the precise subscript of F is inconsequential when computing the residue modulo p over a period. We start as before with the relation $F_i = F_{i-1} + F_{i-2}$ and the derived relation $F_i = 2F_{i-2} + F_{i-3}$. By squaring each of these relations and summing over the period, we obtain

$$a \equiv a + 2b + a \pmod p$$

and

$$a \equiv 4a + 4b + a \pmod p$$

or

$$a + 2b \equiv 0 \pmod p$$

$$4a + 4b \equiv 0 \pmod p$$

Hence we can conclude that a and b must both be congruent to zero modulo p with the possible exception of the case in which the determinant of the coefficients is congruent to zero. But this determinant equals -4 so that the only prime in question would be 2. We find by direct verification that

$$\sum_P F_i^2 \equiv 0 \pmod 2 \text{ but that } \sum_P F_i F_{i-1} \text{ is not.}$$

Theorem 3. With the possible exception of primes 2 and 3 all summations

$$\sum_P F_i^3 \,, \quad \sum_P F_i^2 F_{i-1} \,, \quad \text{and} \quad \sum_P F_i F_{i-1}^2$$

are congruent to zero modulo p.

Proof. We employ the same procedure as before after replacing

$$\sum_P F_i^3 \text{ by } a, \quad \sum_P F_i^2 F_{i-1} \text{ by } b \text{ and } \sum_P F_i F_{i-1}^2 \text{ by } c \,.$$

Starting with

$$F_i = F_{i-1} + F_{i-2}$$

and the two derived relations

$$F_i = 2F_{i-2} + F_{i-3}$$

$$F_i = 3F_{i-3} + 2F_{i-4}$$

we cube each of them and sum over a period to obtain:

$$a \equiv a + 3b + 3c + a \pmod{p}$$
$$a \equiv 8a + 12b + 6c + a \pmod{p}$$
$$a \equiv 27a + 54b + 36c + 8a \pmod{p}$$

or

$$a + 3b + 3c \equiv 0 \pmod{p}$$
$$8a + 12b + 6c \equiv 0 \pmod{p}$$
$$34a + 54b + 36c \equiv 0 \pmod{p}$$

The quantities a, b, and c are all congruent to zero except possibly when the determinant of the coefficients is congruent to zero modulo p. The value of this determinant being $-2^3 3^2$, the only possible exceptions might be the primes 2 and 3.

FURTHER DEDUCTION

It should be noted that if a, b, and c are congruent to zero modulo p, then any expression such as

$$\sum_P F_i^2 F_{i-4}$$

is also congruent to zero modulo p. The reason is that F_{i-4} can be expressed as a linear relation in F_i and F_{i-1} so that this summation becomes a linear combination of a, b, and c. Similar considerations apply for any degree whatsoever. Once it is known that all the summations

$$\sum_P F_i^n, \quad \sum_P F_i^{n-1} F_{i-1}, \quad \sum_P F_i^{n-2} F_{i-1}^2, \cdots, \quad \sum_P F_i^2 F_{i-1}^{n-2}, \quad \sum_P F_i F_{i-1}^{n-1}$$

are all congruent to zero modulo p, then any summation product of degree n of the type we are considering taken over a period will also be congruent to zero modulo p.

GENERAL CASE

The pattern established in the above theorems may clearly be extended to higher degrees. To fix ideas, the fifth power summations will be used. As previously, let $\sum_P F_i^5$ be replaced by a, $\sum_P F_i^4 F_{i-1}$ by b, $\sum_P F_i^3 F_{i-1}^2$ by c, $\sum_P F_i^2 F_{i-1}^3$ by d, and $\sum_P F_i F_{i-1}^4$ by e.

Starting with the relations

$$
\begin{aligned}
F_i &= F_{i-1} + F_{i-2} \\
F_i &= 2F_{i-2} + F_{i-3} \\
F_i &= 3F_{i-3} + 2F_{i-4} \\
F_i &= 5F_{i-4} + 3F_{i-5} \\
F_i &= 8F_{i-5} + 5F_{i-6}
\end{aligned}
$$

we obtain on raising each to the fifth power and summing over a period of the modulus p:

$$
\begin{aligned}
a + 5b + 10c + 10d + 5e &\equiv 0 \pmod{p} \\
2^5 a + 5 \cdot 2^4 b + 10 \cdot 2^3 c + 10 \cdot 2^2 d + 5 \cdot 2e &\equiv 0 \pmod{p} \\
(3^5 + 2^5 - 1)a + 5 \cdot 3^4 2b + 10 \cdot 3^3 2^2 c + 10\, 3^2 2^3 d + 5 \cdot 3 \cdot 2^4 e &\equiv 0 \pmod{p} \\
(5^5 + 3^5 - 1)a + 5 \cdot 5^4 3b + 10 \cdot 5^3 3^2 c + 10 \cdot 5^2 3^3 d + 5 \cdot 5 \cdot 3^4 e &\equiv 0 \pmod{p} \\
(8^5 + 5^5 - 1)a + 5 \cdot 8^4 5b + 10 \cdot 8^3 5^2 c + 10 \cdot 8^2 5^3 d + 5 \cdot 8 \cdot 5^4 e &\equiv 0 \pmod{p}
\end{aligned}
$$

Once again, the quantities a, b, c, d, and e are all congruent to zero modulo p provided:

(1) The determinant of the coefficients is not identically equal to zero; or

(2) The determinant of the coefficients is not congruent to zero modulo p.

Thus precise information on which summations are congruent to zero modulo any given prime is related to knowing the value of the determinant of the coefficients. These determinants have been made the object of extensive study by the author and Terry Brennan who will elaborate the results of their research in a future issue of this publication. For the present, let it suffice to say that the formulas derived empirically by evaluating these determinants to the nineteenth order have now been theoretically justified.

It will be noted that the binomial coefficients of the fifth degree enter into the equations and that these may all be factored from the determinant. As long as the degree of the summation is less than p, these factored binomial coefficients do not affect the issue. Disregarding them, the remaining determinant is as follows.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2^5 & 2^4 & 2^3 & 2^2 & 2 \\ 3^5 + 2^5 - 1 & 3^4 2 & 3^3 2^2 & 3^2 2^3 & 3 \cdot 2^4 \\ 5^5 + 3^5 - 1 & 5^4 3 & 5^3 3^2 & 5^2 3^3 & 5 \cdot 3^4 \\ 8^5 + 5^5 - 1 & 8^4 5 & 8^3 5^2 & 8^2 5^3 & 8 \cdot 5^4 \end{vmatrix}$$

If n be the degree of the summation and the order of the determinant, it is found empirically that:

(1) For $n \equiv 0 \pmod 4$, the value of the determinant is zero. Thus for summations of degree $4k$, none need be congruent to zero modulo any prime.

(2) For $n \equiv 2 \pmod 4$, the value of the determinant is:

$$2 \prod_{i=3}^{n} F_i^{n-i+1} \prod_{i=1}^{n/2} L_i^2 , \tag{1}$$

where L_i indicates the members of the Lucas sequence which is also of the Fibonacci type but with values $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, etc.

(3) For n odd, the value is

$$\prod_{i=3}^{n} F_i^{n-i+1} \prod_{i=1}^{(n+1)/2} L_{2i-1}. \tag{2}$$

For the convenience of the reader the express value of these determinants up to order 20 are given below, omitting those of order $4k$ which are all equal to zero.

VALUE OF DETERMINANT

n	
2	2
3	2^3
5	$2^5\ 3^2\ 5 \cdot 11$
6	$2^{12}\ 3^5\ 5^2$
7	$2^{13}\ 3^4\ 5^3\ 11 \cdot 13 \cdot 29$
9	$2^{24}\ 3^8\ 5^5\ 7^2\ 11 \cdot 13^3\ 17 \cdot 19 \cdot 29$
10	$2^{30}\ 3^{12}\ 5^7\ 7^5\ 11^3\ 13^4\ 17^2$
11	$2^{34}\ 3^{12}\ 5^9\ 7^4\ 11^3\ 13^5\ 17^3\ 19 \cdot 29 \cdot 89 \cdot 199$
13	$2^{52}\ 3^{20}\ 5^{13}\ 7^6\ 11^5\ 13^7\ 17^5\ 19 \cdot 29 \cdot 89^3\ 199 \cdot 233 \cdot 521$
14	$2^{64}\ 3^{30}\ 5^{15}\ 7^9\ 11^7\ 13^9\ 17^6\ 29^3\ 89^4\ 233^2$
15	$2^{73}\ 3^{28}\ 5^{18}\ 7^8\ 11^8\ 13^{11}\ 17^{11}\ 19 \cdot 29^3\ 31 \cdot 61 \cdot 89^5\ 199 \cdot 233^3\ 521$
17	$2^{93}\ 3^{38}\ 5^{24}\ 7^{12}\ 11^{10}\ 13^{15}\ 17^9\ 19 \cdot 29^5\ 31 \cdot 47^2\ 61^3\ 89^7\ 199 \cdot 233^5 \cdot 521 \cdot 1597 \cdot 3571$
18	$2^{111}\ 3^{49}\ 5^{27}\ 7^{16}\ 11^{11}\ 13^{17}\ 17^{11}\ 19^3\ 29^7\ 47^5\ 61^4\ 89^8\ 233^6\ 1597^2$
19	$2^{119}\ 3^{48}\ 5^{30}\ 7^{16}\ 11^{12}\ 13^{19}\ 17^{13}\ 19^3\ 29^7\ 31 \cdot 47^4\ 61^5\ 89^9\ 199 \cdot 233^7\ 521 \cdot 1597^3 \cdot 3571 \cdot 4181 \cdot 9349$

EXAMPLE

For the modulus $p = 19$, it follows from the above determinant values that we might expect to have the sums of powers over a period congruent to zero for $n = 1, 2, 3, 5, 6, 7, 10, 14$. The actual situation is shown in Table 1 from which it is clear that theory is corroborated.

Table 2 shows the powers at which summations of Fibonacci expressions may cease to be congruent to zero modulo p.

Table 3 shows the comparison of theory and calculation for small primes. A 0 in the table indicates by theory and calculation the summation to degree n modulo the given prime is zero; x means that the summation need not be zero by theory; (x) indicates that theory does not require a sum congruent to zero, but that in reality it is congruent to zero. There is in this no contradiction.

Table 1

RESIDUES OF POWERS OF FIBONACCI NUMBERS MODULO 19
(Captions give n)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	4	8	16	13	7	14	9	18	17	15	11	3	6	12	5	10	1
3	9	8	5	15	7	2	6	18	16	10	11	14	4	12	17	13	1
5	6	11	17	9	7	16	4	1	5	6	11	17	9	7	16	4	1
8	7	18	11	12	1	8	7	18	11	12	1	8	7	18	11	12	1
13	17	12	4	14	11	10	16	18	6	2	7	15	5	8	9	3	1
2	4	8	16	13	7	14	9	18	17	15	11	3	6	12	5	10	1
15	16	12	9	2	11	13	5	18	4	3	7	10	17	8	6	14	1
17	4	11	16	6	7	5	9	1	17	4	11	16	6	7	5	9	1
13	17	12	4	14	11	10	16	18	6	2	7	15	5	8	9	3	1
11	7	1	11	7	1	11	7	1	11	7	1	11	7	1	11	7	1
5	6	11	17	9	7	16	4	1	5	6	11	17	9	7	16	4	1
16	9	11	5	4	7	17	6	1	16	9	11	5	4	7	17	6	1
2	4	8	16	13	7	14	9	18	17	15	11	3	6	12	5	10	1
18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
133	114	152	151	152	95	171	111	170	152	127	115	158	95	140	136	126	17

Table 2

p	n odd	n = 4k + 2	p	n odd	n = 4k + 2
2	3	2	43	45	46
3	5	6	47	17	18
5	5	6	53	27	30
7	9	10	59	29	58
11	5	10	61	15	18
13	7	10	67	69	70
17	9	10	71	35	70
19	9	18	73	37	38
23	25	26	79	39	78
29	7	14	83	85	86
31	15	30	89	11	14
37	19	22	97	49	50
41	21	22	101	25	50

Table 3

ZERO AND NON-ZERO SUMMATIONS
FOR SMALL PRIMES

n	5	7	11	13	17	19	23	29	31
3	0	0	0	0	0	0	0	0	0
4	x	x	x	x	x	x	x	x	x
5		0	x	0	0	0	0	0	0
6		0	0	0	0	0	0	0	0
7			x	(x)	0	0	0	x	0
8			x	x	x	x	x	x	x
9			x	(x)	(x)	x	0	x	0
10			x	(x)	(x)	0	0	0	0
11				(x)	(x)	x	0	x	0
12				x	x	x	x	x	x
13					(x)	x	0	x	0
14					(x)	0	0	x	0
15					(x)	x	0	x	x
16					x	x	x	x	x
17						x	0	x	x
18						x	0	x	0
19							0	x	x
20							x	x	x
21							0	x	x
22							0	x	0
23								x	x
24								x	x
25								x	x
26								x	0
27								x	x
28								x	x
29									x
30									x

In addition to the exceptions for $n = 7, 9, 10, 11$ modulo 13 and $n = 9, 10, 11, 13, 14, 15$ modulo 17, an interesting example was found by Dmitri Thoro using a computer. For modulo 199 (period 22), the power summations should be zero for 1, 2, 3, 5, 6, 7, 9, 10, 14, 18 but no others need be zero. Actually, an additional summation congruent to zero was found for $n = 156$.

ADDITIONAL RESEARCH POSSIBILITIES

The following offer additional research possibilities along these lines:

(1) The situation when $n \geq p$.

(2) The theory for composite moduli.

(3) Similar summations for other Fibonacci sequences than F_i .

(4) Possibly by means of additional computer data, the study of cases in which summations are congruent to zero when they need not be; patterns and generalizations in these instances.

LETTER TO THE EDITOR

TWIN PRIMES

Charles Ziegenfus, Madison College, Harrisonburg, Va.

If p and $p + 2$ are (twin) primes, then $p + (p + 2)$ is divisible by 12, where $p > 3$.

Two proofs:

If $p > 3$, then p must be of the form

$$6k + 5 \text{ or } 6k + 1 \quad .$$

If

$$p_{n+1} = p_n + 2 \quad ,$$

then p_n must be of the form $6k + 5$. For otherwise

$$p_{n+1} = (6k + 1) + 2 = 3(2k + 1)$$

and is not prime. Therefore,

$$p_n + p_{n+1} = (6k + 5) + (6k + 5) + 2 = 12(k + 1) \quad .$$

p_n must be of the form $3k$, $3k + 1$, or $3k + 2$. Clearly $p_n = 3k$ since p_n is assumed greater than 3.

If $p_n = 3k + 1$, then $p_{n+1} = 3k + 1 + 2 = 3(k + 1)$ and is not prime. Clearly, $p_n + p_{n+1}$ is divisible by 4.

Now $p_n + p_{n+1} = (3k + 3) + (3k + 2) + 2 = 3(2k + 2)$.

So $p_n + p_{n+1}$ is divisible by 12.