BEGINNERS' CORNER

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THE GOLDEN RATIO: COMPUTATIONAL CONSIDERATIONS

1. INTRODUCTION

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel" — so wrote Kepler (1571—1630) [1].

The famous golden section involves the division of a given line segment into mean and extreme ratio, i. e., into two parts a and b, such that $a/b = b/(a+b)$, $a < b$. Setting $x = b/a$ we have $x^2 - x - 1 = 0$. Let us designate the positive root of this equation by ϕ (the golden ratio). Thus

$$\phi^2 - \phi - 1 = 0 \;. \tag{1}$$

Since the roots of (1) are $\phi = (1+\sqrt{5})/2$ and $-1/\phi = (1-\sqrt{5})/2$ we may write Binet's formula [2], [3], [4] for the n^{th} Fibonacci number in the form

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \tag{2}$$

2. POWERS OF THE GOLDEN RATIO

Returning to (1), let us "solve for ϕ^2" by writing

$$\phi^2 = \phi + 1 \;. \tag{3}$$

Multiplying both members by ϕ, we get $\phi^3 = \phi^2 + \phi = (\phi + 1) + \phi = 2\phi + 1$. Now $\phi^3 = 2\phi + 1$ yields $\phi^4 = 2\phi^2 + \phi = 2(\phi + 1) + \phi = 3\phi + 2$. Similarly,

$$\phi^5 = 3\phi^2 + 2\phi = 3(\phi + 1) + 2\phi = 5\phi + 3 \;.$$

This pattern suggests

$$\phi^n = F_n \phi + F_{n-1} \;, \quad n = 1, 2, 3, \cdots \;. \tag{4}$$

To prove (4) by mathematical induction [5], [6], we note that it is true for $n = 1$ and $n = 2$ (since $F_0 = 0$ by definition). Assume $\phi^k = F_k\phi + F_{k-1}$. Then $\phi^{k+1} = F_k\phi^2 + F_{k-1}\phi = F_k(\phi + 1) + F_{k-1}\phi = (F_k + F_{k-1})\phi + F_k = F_{k+1}\phi + F_k$, which completes the proof. The computational advantage of (4) over expansion of

$$\left(\frac{1 + \sqrt{5}}{2}\right)^n$$

by the binomial theorem is striking.

Dividing both members of (3) by ϕ, we obtain

$$\frac{1}{\phi} = \phi - 1 \;. \tag{5}$$

Thus $1/\phi^2 = 1 - 1/\phi = 1 - (\phi - 1) = -(\phi - 2)$. Using this result and (5), $1/\phi^3 = 2/\phi - 1 = 2(\phi - 1) - 1 = 2\phi - 3$. Similarly, $1/\phi^4 = 2 - 3/\phi = 2 - 3\phi + 3 = -(3\phi - 5)$. Via induction, the reader may provide a painless proof of

$$\phi^{-n} = (-1)^{n+1}(F_n\phi - F_{n+1}) \;, \quad n = 1, 2, 3, \cdots \tag{6}$$

3. A LIMIT OF FIBONACCI RATIOS

If we "solve" $x^2 - x - 1 = 0$ for x by writing $x = 1 + 1/x$ and then consider the related recursion relation

$$x_1 = 1, \quad x_{n+1} = 1 + \frac{1}{x_n} \;, \tag{7}$$

Fibonacci numbers start popping out! We immediately deduce $x_2 = 1 + 1/x_1 = 1 + 1/1 = 2$, $x_3 = 1 + 1/x_2 = 1 + 1/2 = 3/2$, $x_4 = 5/3$, $x_5 = 8/5$, etc. This suggests that $x_n = F_{n+1}/F_n$.

Now suppose the sequence $x_1, x_2, x_3, \cdots$ has a limit, say L, as $n \to \infty$. Then

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = L$$

whence (7) yields $L = 1 + 1/L$ or $L = \phi$ since the x_i are positive. Indeed, there are many ways of proving Kepler's observation that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi \quad . \tag{8}$$

E.g., from (2)

$$\frac{F_{n+1}}{F_n} = \left[\phi^{n+1} - (-\phi)^{-n-1}\right] \Big/ \left[\phi^n - (-\phi)^{-n}\right] = \frac{\phi - \dfrac{1}{(-\phi)^{n+1}\phi^n}}{1 - \dfrac{1}{(-\phi)^n\phi^n}} \to \phi$$

since $\phi = (1 + \sqrt{5})/2 > 1$
implies that the fractions involving ϕ^n approach 0 as $n \to \infty$.

4. AN APPROXIMATE ERROR ANALYSIS

Just how accurate are the above approximations to the golden ratio? Let us denote the exact error at the n^{th} iteration by

$$e_n \equiv x_n - \phi \tag{9}$$

The trick is to express e_{n+1} in terms of e_n and then to make use of the identity

$$\frac{1}{1 + w} = 1 - w + w^2 - w^3 + w^4 - \cdots , \quad w < 1 \quad . \tag{10}$$

(The latter may be discovered by dividing 1 by $1 + w$; cf. [7].)

Thus

$$
\begin{aligned}
e_{n+1} \equiv x_{n+1} - \phi &= 1 + \frac{1}{x_n} - \phi \\
&= 1 - \phi + \frac{1}{e_n + \phi} = 1 - \phi + \frac{1}{\phi}\left[\frac{1}{1 + (e_n/\phi)}\right] \\
&= 1 - \phi + \frac{1}{\phi}\left[1 - (e_n/\phi) + (e_n/\phi)^2 - (e_n/\phi)^3 + \cdots\right] \\
&= -\frac{e_n}{\phi^2} + \frac{e_n^2}{\phi^3} - \frac{e_n^3}{\phi^4} + \cdots \quad \text{since } \frac{1}{\phi} = \phi - 1 \text{ by (5)} .
\end{aligned}
$$

However, the terms involving the higher powers of e_n are quite small in comparison with the first term. Thus, following the customary practice of neglecting high order terms, we will <u>approximate</u> the error at the $(n + 1)$st step by $\epsilon_{n+1} = -\epsilon_n \phi^{-2}$. Finally, we may note that $\epsilon_2 = -\epsilon_1 \phi^{-2}$, $\epsilon_3 = -\epsilon_2 \phi^{-2} = \epsilon_1 \phi^{-4}$, $\epsilon_4 = -\epsilon_1 \phi^{-6}$, and, in general,

$$
\epsilon_n = (-1)^{n+1} \epsilon_1 \phi^{-2(n-1)} . \tag{11}
$$

5. COMPUTATION OF ϕ VIA MATRICES

We recall (cf. [8]) that if the matrix

$$
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

and the column vector

$$
v = \begin{pmatrix} r \\ s \end{pmatrix}
$$

then the product Mv is defined to be the column vector

$$\begin{pmatrix} ar + bs \\ cr + ds \end{pmatrix} .$$

Let us investigate the recursion relation

$$v_{n+1} = A v_n , \quad n = 1, 2, 3, \cdots \tag{12}$$

where A is a given matrix and v_1 a given vector. (For convenience we will always take v_1 to be the first column of A.)

(a) If A is the Q matrix [9], [10] $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = Av_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $v_3 = Av_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $v_4 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $v_5 = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$, $\cdots$, $v_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$, $\cdots$. Thus if $v_i = \begin{pmatrix} r_i \\ s_i \end{pmatrix}$, then for $A = Q$ the ratio r_i / s_i is precisely the approximation to ϕ obtained from (7).

(b) Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then

$$v_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 34 \\ 21 \end{pmatrix}, \cdots$$

This time $v_n = \begin{pmatrix} F_{2n+1} \\ F_{2n} \end{pmatrix}$. Note that here the ratio obtained from, say, v_3 is exactly that obtained from v_6 when A is taken to be the Q matrix.

(c) For $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, the successive approximations suggested by (12) turn out to be

$$\frac{1}{-1}, \ \frac{-1}{2}, \ \frac{2}{-3}, \ \frac{-3}{5}, \ \frac{5}{-8}, \ \cdots \tag{13}$$

From the discussion in (a) above it is easy to deduce that the limit of the sequence (13) is $-\frac{1}{Q}$: the negative root of (1)!

Similarly pleasant results may be obtained from (infinitely many) other A's. Several possibilities are suggested in the following exercises. The mathematical basis for this approach will be explored in a future issue.

6. EXERCISES

E1. Show that the definition of the golden section leads to the equation $x^2 - x - 1 = 0$.

E2. Use mathematical induction to prove (6).

E3. How should you define F_{-k} $(k > 0)$ in order that (4) would hold for negative values of n?

E4. Verify (10) by long division. Find an additional check by starting with the right-hand member.

E5. Give an induction proof of (11).

E6. Show that when $x_1 = 1$, the estimated error given by (11) becomes

$$\epsilon_n = (-1)^n \phi^{1-2n}$$

Hint: Use (5).

E7. Using the results of E6 (with $\phi = 1.618$) compute an estimate of $F_{11}/F_{10} - \phi$. Compare this approximate error to the actual error (given $\phi = 1.61803$). Thus although ϵ_n is a function of ϕ itself, it can be used in approximating ϕ to a desired number of decimal places.

E8. A comparison of the three values of A exhibited above reveals that in each case A has the form

$$\begin{pmatrix} w & 1 \\ 1 & w - 1 \end{pmatrix}.$$

It turns out that w need not be an integer. Experiment with different values of w. Hint: consider the cases

(a) $w > \phi$

(b) $\frac{1}{2} < w < \phi$

(c) $$-\frac{1}{\phi} < w < \frac{1}{2}$$

(d) $$w < -\frac{1}{\phi}$$

E9. What happens, in the preceding exercise, when $w = 1/2$?

E10. Explain why the first two illustrations of (12) are essentially "computationally equivalent." Hint: count the minimum number of arithmetic operations required in each case.

REFERENCES

1. H. S. M. Coxeter, Introduction to Geometry, John Wiley and Sons, Inc., New York, 1961, p. 160.
2. The Fibonacci Quarterly, 2 (1963), 66–67.
3. __________, 2 (1963), 73.
4. __________, 2 (1963), 75.
5. __________, 1 (1963), 61.
6. __________, 1 (1963), 67.
7. __________, 2 (1963), 65.
8. __________, 2 (1963), 61.
9. __________, 2 (1963), 47.
10. __________, 2 (1963), 61–62.

EDITORIAL REMARK FROM PAGE 19:

In the European notation, 763,26 means what 763.26 does to us, and 2.5^x means $2(5^x) = 2 \cdot 5^x$.

HAVE YOU SEEN?

Nathan J. Fine, Generating Functions, Enrichment Mathematics for High School, Twenty-eighth Yearbook National Council of Teachers of Mathematics, Washington, D.C., 1963, pp. 355–367. This is an excellent and inspiring article.