# ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY S. L. BASIN, SYLVANIA ELECTRONIC SYSTEMS, MT. VIEW, CALIF.

Send all communications regarding Elementary Problems and Solutions to S. L. Basin, 946 Rose Ave., Redwood City, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer must submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated. Solutions should be submitted within two months of the appearance of the problems.

B-24 *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

It is evident that the determinant

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+2} & F_{n+3} & F_{n+4} \end{vmatrix}$$

has a value of zero. Prove that if the same quantity $k$ is added to each element of the above determinant, the value becomes $(-1)^{n-1}k$.

B-25 *Proposed by Brother U. Alfred.*

Find an expression for the general term(s) of the sequence $T_0 = 1, T_1 = a, T_2 = a, \cdots$ where

$$T_{2n} = \frac{T_{2n-1}}{T_{2n-2}} \quad \text{and} \quad T_{2n+1} = T_{2n}T_{2n-1} \quad .$$

B-26 *Proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, Calif.*

Given polynomials $b_n(x)$ and $B_n(x)$ defined by

$$b_0(x) = 1, \quad B_0(x) = 1$$

$$b_n(x) = xB_{n-1}(x) + b_{n-1}(x) \quad (n \geq 1)$$

$$B_n(x) = (x+1)B_{n+1}(x) + b_{n-1}(x) \quad (n \geq 1)$$

show that

$$b_n(x) = P_{2n}(x)$$

and

$$B_n(x) = P_{2n+1}(x)$$

where

$$P_m(x) = \sum_{j=0}^{\left[\frac{m}{2}\right]} \binom{n-j}{j} x^{\left[\frac{m}{2}\right]-j}$$

$\left[\frac{m}{2}\right]$ being the greatest integer less than or equal to $\frac{m}{2}$ .

B-27 *Proposed by D.C. Cross, Birmingham, England.*

Let $x = \cos\phi$, $[z]$ is the greatest integer contained in $z$ .

$$\begin{aligned}
\cos\phi &= x \\
\cos 2\phi &= 2x^2 - 1 \\
\cos 3\phi &= 4x^3 - 3x \\
\cos 4\phi &= 8x^4 - 8x^2 + 1 \\
\cos 5\phi &= 16x^5 - 20x^3 + 5x \\
\cos 6\phi &= 32x^6 - 48x^4 + 18x^2 - 1
\end{aligned}$$

$$\cos n\phi = P_n(x) = \sum_{j=1}^{N} A_{jn} X^{n+2-2j}$$ ($N = [(n+1)/2]$ is greatest integer function.)

Show

(i) $A_{1n} = 2^n$

(ii) $A_{j+1,n+1} = 2A_{j+1,n} - A_{j,n}$ $(j = 1, 2, \cdots N-1)$

(iii) $P_{n+2}(x) = 2x P_{n+1}(x) - P_n(x)$

(iv) If $A_n = \sum_{j=1}^{N} |A_{jn}|$, then $A_{n+2} = 2A_{n+1} + A_n$ .

Note: ($A_1 = 1$, $A_2 = 3$, $7 = A_3 = 2A_2 + A_1 = 2 \cdot 3 + 1$).

B-28 *Proposed by Brother U. Alfred.*

Using the nine Fibonacci numbers $F_2$ to $F_{10}$ (1, 2, 3, 5, 8, 13, 21, 34, 55), determine a third-order determinant having each of these numbers as elements so that the value of the determinant is a maximum.

B-29 *Proposed by A.P. Boblett, U.S. Naval Ordnance Laboratory, Corona, California.*

Define a general Fibonacci sequence such that

$$F_1 = a; \quad F_2 = b; \quad F_n = F_{n-2} + F_{n-1} \quad n \geq 3$$
$$F_n = F_{n+2} - F_{n+1} \quad n \leq 0$$

Also define a characteristic number, C, for this sequence, where $C = (a + b)(a - b) + ab$.

Prove:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n C, \quad \text{for all } n \quad .$$

SOLUTIONS

Solutions to Problems B6 and B9 through B15, Vol. 1, No. 2, April, 1963

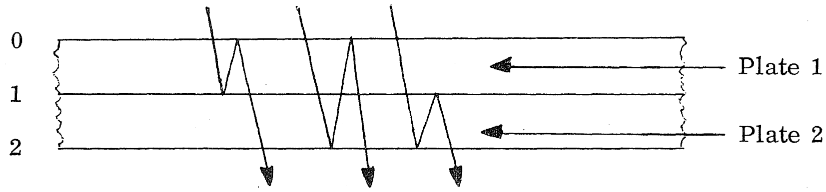
SOME REFLECTIONS

B-6 *Proposed by Leo Moser, University of Alberta, Edmonton, Alberta.*

Light rays fall upon a stack of two parallel plates of glass, one ray goes through without reflection, two rays (one from each interval interface opposing the ray) will be reflected once but in different ways, three will be reflected twice but in different ways. Show that the number of distinct paths, which are reflected exactly n times, is $F_{n+2}$.

Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania

All rays which experience exactly n reflections will emerge from the same face, either top or bottom of the stack; furthermore, if those having n - 1 reflections emerge from the top face, then those having n reflections will emerge from the bottom face. Let us assume, without loss of generality that the rays having exactly n reflections will emerge from the bottom face as shown below for the case of two reflections.



Let $\alpha_n$ be the number of distinct paths which have exactly n reflections. If we consider any emergent ray which has had n reflections ($n \geq 2$), then it must have had its last, or $n^{th}$ reflection from either face 0 or interface 1. The number of distinct paths having the $n^{th}$ reflection at face 0 is equal to the number of distinct paths reaching face 0 after n - 1 reflections, or $\alpha_{n-1}$. Similarly, the paths whose $n^{th}$ reflection is at interface 1 must have had the (n - 1)th reflection at face 2, and the number of distinct paths is then equal to the number of distinct paths reaching face 2 after (n - 2) reflections, or $\alpha_{n-2}$. Since the two possibilities are mutually exclusive and exhaustive, we have $\alpha_n = \alpha_{n-1} + \alpha_{n-2}$ for $n \geq 2$. The initial conditions, $\alpha_0 = 1$, $\alpha_1 = 2$ establish that $\alpha_n = F_{n+2}$ for $n \geq 0$.

## FIBONACCI SUMS

B-9 *Proposed by R.L. Graham, Bell Telephone Laboratories, Murray Hill, N.J.*

Prove

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = 1$$

and

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = 2 \quad ,$$

where $F_n$ is the $n^{th}$ Fibonacci number.

B-9 Solution by Francis D. Parker, University of Alaska.

Since

$$\frac{1}{F_{n-1}F_{n+1}} = \frac{F_n}{F_{n-1}F_nF_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_nF_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}$$

then

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}} = \sum_{n=2}^{\infty}\left[\frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}\right] = \left[\frac{1}{1\cdot 1} - \frac{1}{1\cdot 2}\right] + \left[\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right] + \left[\frac{1}{2\cdot 3} - \frac{1}{3\cdot 5}\right] + \cdots = 1$$

Similarly,

$$\frac{F_n}{F_{n-1}F_{n+1}} = \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}} - \frac{1}{F_{n+1}} \quad \text{and}$$

and

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{1} - \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{1}{5}\right] + \left[\frac{1}{3} - \frac{1}{8}\right] + \cdots = \frac{1}{F_1} + \frac{1}{F_2} = 2$$

Editorial Comment: The above solution to problem B-9 is a good example of a principle found in many other problems in number theory, namely in forming a sum, it is often helpful to judiciously group the terms in a certain fashion. An example of this may be found in proving the following theorem concerning the divisor function $\tau(n)$. Prove $\tau(n)$ is odd if and only if n is a square.

LUCAS-FIBONACCI IDENTITY

B-10 *Proposed by Stephen Fisk, San Francisco, California.*

Prove the "de Moivre-type" identity,

$$\left(\frac{L_n + \sqrt{5}\,F_n}{2}\right)^p = \frac{L_{np} + \sqrt{5}\,F_{np}}{2}$$

where $L_n$ denotes the nth Lucas number and $F_n$ denotes the nth Fibonacci number.

B-10 Solution by Charles Wall, Ft. Worth, Texas.

Since

$$\frac{L_n + \sqrt{5}\,F_n}{2} = \frac{\alpha^n + \beta^n + \alpha^n - \beta^n}{2} ,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} ,$$

we have

$$\left(\frac{L_n + \sqrt{5}\,F_n}{2}\right)^p = \alpha^{np} = \frac{\alpha^{np} + \beta^{np} + \alpha^{np} - \beta^{np}}{2} = \frac{L_{np} + \sqrt{5}\,F_{np}}{2} .$$

B-11 *Proposed by S.L.Basin, Sylvania Electronic Defense Laboratory,*

Show that the hypergeometric function

$$G(x,n) = \sum_{k=0}^{n-1} \frac{2^k (n+k)!\,(x-1)^k}{(n-k-1)!\,(2k+1)!}$$

generates the sequence

$$G\left(\frac{3}{2}, n\right) = F_{2n}, \qquad n = 1,2,3,\cdots .$$

B-11 Solution by S. L. Basin, Sylvania Electronic Systems, Mountain View, California and San Jose State College

$$\sum_{k=0}^{n-1} \frac{2^k (n+k)!\,(x-1)^k}{(n-k-1)!\,(2k+1)!} = U_{n-1}(x)$$

where $U_{n-1}(x)$ are the Chebyshev polynomials of the second kind and

$$U_{n-1}(x) = \frac{1}{2\sqrt{x^2 - 1}} \left\{(x + \sqrt{x^2 - 1})^n - (x - \sqrt{x^2 - 1})^n\right\}$$

$$U_{n-1}\left(\frac{3}{2}\right) = \frac{1}{\sqrt{5}} \left\{ \left(\frac{3 + \sqrt{5}}{2}\right)^n - \left(\frac{3 - \sqrt{5}}{2}\right)^n \right\}$$

Observing that

$$\left(\frac{3 + \sqrt{5}}{2}\right) = \left(\frac{1 + \sqrt{5}}{2}\right)^2 \text{ and } \left(\frac{3 - \sqrt{5}}{2}\right) = \left(\frac{1 - \sqrt{5}}{2}\right)^2 ,$$

we have

$$U_{n-1}\left(\frac{3}{2}\right) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1 + \sqrt{5}}{2}\right)^{2n} - \left(\frac{1 - \sqrt{5}}{2}\right)^{2n}\right\} = F_{2n} .$$

Comment: Setting $x = 3/2$, the summation becomes

$$\sum_{k=0}^{n-1} \frac{(n + k)!}{(2k + 1)!\ (n - k - 1)!} = \sum_{k=0}^{n-1} \binom{n + k}{2k + 1} = F_{2n} \quad \left(\begin{matrix}\text{Rising diagonals} \\ \text{of Pascal's} \\ \text{triangle}\end{matrix}\right)$$

See Fig. 1, page 24, October, 1963, Fibonacci Quarterly.

## A LUCAS DETERMINANT

B-12 *Proposed by Paul F. Byrd, San Jose State College, San Jose, Calif.*

Show that

$$L_{n+1} = \begin{vmatrix} 3 & i & 0 & 0 & \cdots & 0 & 0 \\ i & 1 & i & 0 & \cdots & 0 & 0 \\ 0 & i & 1 & i & \cdots & 0 & 0 \\ 0 & 0 & i & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 & i \\ 0 & 0 & 0 & 0 & \cdots & i & 1 \end{vmatrix}_n \qquad n \geq 1$$

where $L_n$ is the nth Lucas number given by $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$, and $i = \sqrt{-1}$ .

B-12 Solution by Marjorie Bicknell, San Jose State College, San Jose, Calif.

Let $D_n$ denote the determinant of order n. Expanding the determinant by its nth row we have, $D_n = D_{n-1} + D_{n-2}$ with $D_1 = 3$, $D_2 = 4$ so that $D_n = L_{n+1}$.

Also solved by William A. Beyer, Los Alamos, New Mexico

## FIBONACCI CONTINUANT

B-13 *Proposed by S.L. Basin.*

Determinants of order n which are of the form,

$$K_n(b,c,a) = \begin{vmatrix} c & a & 0 & 0 & 0 & \cdots \\ b & c & a & 0 & 0 & \cdots \\ 0 & b & c & a & 0 & \cdots \\ 0 & 0 & b & c & a & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n$$

are known as <u>CONTINUANTS</u>

Prove that

$$K_n(b,c,a) = \frac{(c + \sqrt{c^2 - 4ab})^{n+1} - (c - \sqrt{c^2 - 4ab})^{n+1}}{2^{n+1}\sqrt{c^2 - 4ab}}$$

and show, for special values of a, b, and c, that $K_n(b,c,a) = F_{n+1}$ .

B-13 Solution by Marjorie Bicknell, San Jose State College, San Jose, Calif.

Expanding $K_n(b,c,a)$ by the nth row we obtain,

(1) $$K_n(b,c,a) = cK_{n-1}(b,c,a) - ab\,K_{n-2}(b,c,a) \quad .$$

If u and v are the roots of the quadratic equation $x^2 - cx + ab = 0$, then

(2) $$u = \frac{1}{2}(c + \sqrt{c^2 - 4ab}), \quad v = \frac{1}{2}(c - \sqrt{c^2 - 4ab})$$

Now $K_n(b,c,a) = (u^{n+1} - v^{n+1})/(u - v)$ by induction and $K_n(b,c,a) = F_{n+1}$ for values of a, b, and c which yield the quadratic $x^2 - x - 1$, i.e., $a = c = 1$, and $b = -1$; $a = -1$ and $b = c = 1$; $a = b = i = \sqrt{-1}$ and $c = 1$.

## A LITTLE SURPRISE

B-14 *Proposed by Maxey Brooke, Sweeny, Texas and C.R. Wall, Ft. Worth, Tex*

Show that

$$\sum_{n=1}^{\infty} \frac{F_n}{10^n} = \frac{10}{89} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} F_n}{10^n} = \frac{10}{109} \quad .$$

B-14 Solution by Charles Wall, Ft. Worth, Texas

Since
$$\sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

then

$$\sum_{n=1}^{\infty} F_n(.1)^n = \frac{.10}{1 - .10 - .01} = \frac{.10}{.89} = \frac{10}{89}$$

and

$$\sum_{n=1}^{\infty} [-F_n(-.1)^n] = \frac{-(-.10)}{1 + .10 - .01} = \frac{.10}{1.09} = \frac{10}{109} .$$

Also solved by Dermott A. Breault, Sylvania, ARL, Waltham, Mass.

## FIBONACCI SEQUENCE PERIODS

B-15 *Proposed by R.B.Wallace, Beverly Hills, Calif., and Stephen Geller, University of Alaska, College, Alaska.*

If $p_k$ is the smallest positive integer such that

$$F_{n+p_k} \equiv F_n \bmod (10^k)$$

for all positive $n$, then $p_k$ is called the period of the Fibonacci sequence relative to $10^k$. Show that $p_k$ exists for each $k$, and find a specific formula for $p_k$ as a function of $k$.

Editorial Comment: This problem is discussed in this issue in a paper by Dov Jarden which is a reply to Stephen Geller's letter to the editor, p. 84, April, 1963, Fibonacci Quarterly.

EDITORIAL ASSOCIATES (Cont.)

well as those who have the intention of doing so, will receive recognition as Editorial Associates. The Editor should be contacted by anyone who wishes to be associated with the Fibonacci Quarterly in this manner.

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