# LINEAR RECURRENCE RELATIONS — PART II

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## 1. INTRODUCTION

By applying the exponential generating function transformation

$$(1.1) \qquad Y(t) = \sum_{n=0}^{\infty} y_n \frac{t^n}{n!} ,$$

we derived in Part I of this article [1] an explicit formula for the general solution of the homogeneous linear recurrence relation

$$(1.2) \qquad L_k(E)\, y_n \equiv \sum_{j=0}^{k} a_j E^j y_n \equiv \sum_{j=0}^{k} a_j y_{n+j} = 0 ,$$

where the coefficients $a_j$ were constants, and the translation operator $E^j$ was defined by

$$E^j y_n = y_{n+j} \quad (j = 0, 1, \cdots, k) .$$

In the present part of this article, we discuss the non-homogeneous recurrence relations having variable coefficients.

## 2. EXPLICIT SOLUTION OF A NON-HOMOGENEOUS RECURRENCE RELATION

We consider the linear non-homogeneous recurrence relation

$$(2.1) \qquad \sum_{j=0}^{k} a_j y_{n+j} \equiv L_k(E)\, y_n = b_n$$

with constant coefficients, and where the roots $r_1, r_2, \cdots, r_k$ of the characteristic equation $L_k(r) = 0$ are all distinct. Multiplying both sides of (2.1) by $t^n/n!$ and summing over $n$ from $0$ to $\infty$ yield the transformed equation

$$(2.2) \qquad L_k(D)\, Y = B(t) , \qquad \left( D \equiv \frac{d}{dt} \right)$$

where

$$B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \quad . \tag{2.3}$$

Now (2.2) is an ordinary linear differential equation whose general solution is

$$Y(t) = Y_p(t) + \sum_{i=1}^{k} c_i e^{r_i t} \quad , \tag{2.4}$$

where, by the method of variation of parameters, the particular solution $Y_p(t)$ can be expressed by

$$Y_p(t) = \sum_{i=1}^{k} \frac{e^{r_i t}}{L_k'(r_i)} \sum_{n=0}^{\infty} \frac{b_n}{n!} \int_0^t s^n e^{-r_i s}\, ds \tag{2.5}$$

or

$$Y_p(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{i=1}^{k} \frac{r_i^n}{L_r'(r_i)} \sum_{p=0}^{n-1} \frac{b_p}{r_i^{p+1}} \quad . \tag{2.6}$$

Since $y_n = Y^{(n)}(0)$, we immediately find that

$$y_n = \sum_{i=1}^{k} c_i r_i^n + \sum_{i=1}^{k} \frac{r_i^n}{L_k'(r_i)} \sum_{p=0}^{n-1} \frac{b_p}{r_i^{p+1}} \tag{2.7}$$

is the general solution of the recurrence relation (2.1). The case where $L_k(r) = 0$ has repeated roots may be treated in a similar way and is left to the reader.

## 3. LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

A generalization of the recurrence relation (1.2) with constant coefficients is the equation

$$\sum_{j=0}^{k} P_j(n)\, y_{n+j} = 0 \quad , \tag{3.1}$$

where $P_j(n)$ are polynomials of degree $q_j$ in the independent discrete variable n. If the exponential generating function (1.1) is applied to (3.1), we obtain the differential equation

$$(3.2)\qquad \sum_{j=0}^{k} P_j(\phi)\, Y^{(j)} = 0$$

where $\phi$ is the operator

$$(3.3)\qquad \phi \equiv t\ D \equiv t\frac{d}{dt} \quad ,$$

and where, by definition,

$$(3.4)\qquad P_j(n) = \sum_{m=0}^{q_j} \alpha_m n^m \quad .$$

Equation (3.2) is an immediate consequence of the following theorem which can easily be established by mathematical induction:

Theorem 3.1. The exponential generating function for the sequence $\{n^m y_{n+j}\}$ is given by

$$(3.5)\qquad \phi^m Y^{(j)}(t) = \sum_{n=0}^{\infty} n^m y_{n+j} \frac{t^n}{n!} \quad , \quad (j = 1,2,\cdots;\ m = 0,1,\cdots,)$$

where $\phi$ is defined by (3.3).

Since the theory of differential equations is richer in special formulas and techniques than the corresponding formulas and techniques in the theory of recurrence relations, equation (3.2) resulting from the application of the exponential generating function may be more amenable to an explicit solution than the original relation (3.1). We illustrate this fact with the following examples:

## 4. EXAMPLES WITH VARIABLE COEFFICIENTS

The Bessel functions $J_n(x)$ of order $n$ satisfy the recurrence relation

$$(4.1)\qquad x y_{n+2}(x) - 2(n + 1) y_{n+1}(x) + x y_n(x) = 0 \quad ,$$

which is a very special case of (3.1) with $k = 2$, $P_2(n) = x$, $P_1(n) = -2(n + 1)$, $P_0(n) = x$. Equation (3.2) thus yields the differential equation

$$(4.2)\qquad (x - 2t) Y'' - 2 Y' + x Y = 0 \quad ,$$

which has the particular solution

(4.3) $$Y = J_0(\sqrt{x^2 - 2tx}) \quad ,$$

where $J_0(z)$ is Bessel's function of zero order defined by [2]

(4.4) $$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{4^m (m!)^2} \quad .$$

Thus, we find

$$\begin{aligned} Y = J_0(\sqrt{x^2 - 2tx}) &= \sum_{m=0}^{\infty} \frac{(-1)^m (x^2 - 2tx)^m}{4^m (m!)^2} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{4^m (m!)^2} \sum_{n=0}^{m} \binom{m}{n} (-1)^n \left(\frac{2t}{x}\right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2t}{x}\right)^n \sum_{m=n}^{\infty} \frac{(-1)^m}{4^m} \binom{m}{n} \frac{x^{2m}}{(m!)^2} \quad , \end{aligned}$$

or finally

(4.5) $$Y = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+n}}{2^{2j+n} j! (j+n)!} \quad .$$

By definition of the generating function (1.1), we therefore have

(4.6) $$y_n(x) \equiv J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+n}}{2^{2j+n} j! (j+n)!} \quad .$$

As a final example, we consider the second-order recurrence relation

(4.7) $$y_{n+2}(x) - 2x y_{n+1}(x) + 2(n+1) y_n(x) = 0$$

which is satisfied by the Hermite polynomials $H_n(x)$ of degree $n$, with initial values

(4.8) $$y_0(x) = 1, \quad y_1(x) = 2x \quad .$$

The transformed equation of relation (4.6) is the differential equation

$$Y'' - 2(x - t)Y' + 2Y = 0 \tag{4.9}$$

with conditions $Y(0,x) = 1$ and $Y'(0,x) = 2x$. Solution of (4.8) is

$$Y(t,x) = e^{x^2} \cdot e^{-(x-t)^2} = e^{2tx-t^2}, \tag{4.10}$$

and expansion of the right side thus yields

$$Y = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{[n/2]} \frac{(-1)^m (2x)^{n-2m}}{m!\,(n-2m)!}, \tag{4.11}$$

where $[n/2]$ means the integral part $n/2$. From the definition of the exponential generating function (1.1), it is seen that

$$y_n \equiv H_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m n!\,(2x)^{n-2m}}{m!\,(n-2m)!} \tag{4.12}$$

is the explicit solution of the recurrence relation (4.6)

## 5. REMARKS

The Laguerre polynomials, and in fact most of the important special functions of mathematical physics, satisfy a second-order recurrence relation of the form

$$[A_2(x) + nB_2(x)]y_{n+2}(x) + [A_1(x) + nB_1(x)]y_{n+1}(x) + [A_0(x) + nB_0(x)]y_n(x) = 0 \tag{5.1}$$

whose coefficients are linear in the independent real variable $n$. Explicit solutions for them, by the method of generating functions, may be obtained as in the above two examples. The method of generating functions can also be easily applied to solve certain <u>partial</u> recurrence relations. In part III of this article we shall show how this may be done and give examples of solutions involving Fibonacci arrays.

## REFERENCES

See page 34 for the references to this article.