8. AN EXTENDED RESULT

Theorem 5. The series

$$A = \sum_{m=1}^{\infty} (-1)^{m+1} \operatorname{Tan}^{-1} \frac{1}{F_{2m}}$$

converges and $A = \operatorname{Tan}^{-1} (\sqrt{5} - 1)/2$.

Proof: Since the series is an alternating series, and, since $\operatorname{Tan}^{-1} X$ is a continuous increasing function, then

$$\operatorname{Tan}^{-1} \frac{1}{F_{2n}} > \operatorname{Tan}^{-1} \frac{1}{F_{2n+2}} \text{ and } \operatorname{Tan}^{-1} 0 = 0 .$$

The angle A must lie between the partial sums S_N and S_{N+1} for every $N > 2$ by the error bound in the alternating series, but $S_N = \operatorname{Tan}^{-1} (F_N / F_{N+1})$. Thus the angles of U_N and U_{N+1} lie on opposite sides of A. By the continuity of $\operatorname{Tan}^{-1} X$ then

$$\lim_{n\to\infty} \operatorname{Tan}^{-1} (F_n / F_{n+1}) = A = \operatorname{Tan}^{-1} (\sqrt{5} - 1)/2 .$$

Comment: The same result can be obtained simply from

$$\operatorname{Tan} \left\{ \operatorname{Tan}^{-1} \frac{F_n}{F_{n+1}} - \frac{\sqrt{5} - 1}{2} \right\} = (-1)^{n+1} \left(\frac{\sqrt{5} - 1}{2} \right)^{2n+1} .$$

Which slope gives a better numerical approximation to $\frac{\sqrt{5} - 1}{2}$, F_n / F_{n+1} or L_n / L_{n+1}? Hmmm?

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