SOME REMARKS ON CARLITZ' FIBONACCI ARRAY

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Recently in this journal [Vol. 1, No. 2, pp. 17—27] Carlitz defined a Fibonacci array. Among the properties not included in his discussion are the following summation formulas: (Recall $u_{0,n} = F_n$; $u_{1,n} = F_{n+2}$; $u_{r,n} = u_{r-1,n} + u_{r-2,n}$)

(I) $$\sum_{n=0}^{r} u_{r-n,n} = \frac{2}{5}\left[(r+1)L_{r+1} - F_{r+1}\right] ,$$

(II) $$\sum_{n=0}^{r} (-1)^n u_{r-n,n} = 0 ,$$

(III) $$\sum_{n=0}^{r} \binom{r}{n} u_{r-n,n} = \frac{1}{5}\left[2^{r+1}L_{r+1} - 2\right] ,$$

(IV) $$\sum_{n=0}^{r} (-1)^{n+1} \binom{r}{n} u_{r-n,n} = \begin{cases} 0 & \text{if } r \text{ odd or } r = 0 \\ 2 \cdot 5^{(r-2)/2} & \text{if } r/2 \in J^+ \end{cases} .$$

The similarities between the formulas above and the four below should be noted:

$$\sum_{n=0}^{r} F_n F_{r-n} = \frac{1}{5}\left[rL_r - F_r\right] ,$$

$$\sum_{n=0}^{r} (-1)^{n+1} F_n F_{r-n} = \begin{cases} 0 & \text{if } r \text{ odd} \\ F_r & \text{if } r \text{ even} \end{cases} ,$$

$$\sum_{n=0}^{r} \binom{r}{n} F_n F_{r-n} = \frac{1}{5}\left[2^r L_r - 2\right] ,$$

$$\sum_{n=0}^{r} (-1)^{n+1} \binom{r}{n} F_n F_{r-n} = \begin{cases} 0 & \text{if } r \text{ odd or } r = 0 \\ 2 \cdot 5^{(r-2)/2} & \text{if } r/2 \in J^+ \end{cases} .$$

Because of an overabundance of properties in Carlitz' discussion, we may generalize his array in two ways, taking $H_1 = p$, $H_2 = p + q$,

$$H_{n+1} = H_n + H_{n-1} \quad .$$

We make no attempt to generalize all his results, but consider only the simpler ones. Arabic numerals referring to formulas correspond to those in Carlitz' article.

I. FIRST GENERALIZATION

We define

$$G_{0,n} = H_n \quad , \tag{1'}$$

$$G_{1,n} = H_{n+2} \quad , \tag{2'}$$

as the first two rows of the generalized array G. For $r > 1$ we define $G_{r,n}$ by means of

$$G_{r,n} = G_{r-1,n} + G_{r-2,n} \quad . \tag{3'}$$

It follows that

$$G_{r,n} = pu_{r,n} + qu_{r,n-1}$$

and

$$G_{r,n} = G_{r,n-1} + G_{r,n-2} \quad . \tag{4'}$$

From these properties Table I is easily computed.

Table I
ARRAY G

r \ n	0	1	2	3	4
0	q	p	p + q	2p + q	3p + 2q
1	p + q	2p + q	3p + 2q	5p + 3q	8p + 5q
2	p + 2q	3p + q	4p + 3q	7p + 4q	11p + 7q
3	2p + 3q	5p + 2q	7p + 5q	12p + 7q	19p + 12q
4	3p + 5q	8p + 3q	11p + 8q	19p + 11q	30p + 19q

The symmetry property (5) obviously fails since $G_{0,1} \neq G_{1,0}$. If we put

$$(6')\qquad g_r(x) = \sum_{n=0}^{\infty} G_{r,n} x^n$$

we find that

$$(7')\qquad g_0(x) = \frac{q + px - qx}{1 - x - x^2} , \quad g_1(x) = \frac{p + q + px}{1 - x - x^2} .$$

We also have

$$(8')\qquad g_r(x) = g_{r-1}(x) + g_{r-2}(x) ,$$

so that

$$(9')\qquad g_r(x) = \frac{H_r + xH_{r+1} + q(F_r - xF_{r+1})}{1 - x - x^2} .$$

Putting

$$g(x,y) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} G_{r,n} x^r y^n$$

we have

$$g(x,y) = \sum_{r=0}^{\infty} \frac{H_r + yH_{r+1} + q(F_r - yF_{r+1})}{1 - y - y^2} x^r$$

so that

$$(11')\qquad g(x,y) = \frac{px + py + q - qy + qxy}{(1 - x - x^2)(1 - y - y^2)} .$$

It appears that

$$(13')\qquad \begin{cases} G_{r+1,r-1} - G_{r,r} = (-1)^r (p - q) \\ G_{r-1,r+1} - G_{r,r} = (-1)^r p \end{cases} .$$

Indeed, following Carlitz' procedure we find that

$$(14')\qquad \begin{cases} G_{r+2,r-2} - G_{r,r} = (-1)^{r+1}(p - 2q) \\ G_{r-2,r+2} - G_{r,r} = (-1)^{r+1}(p + q) \end{cases},$$

$$(15')\qquad \begin{cases} G_{r+3,r-3} - G_{r,r} = (-1)^{r}(4p - 6q) \\ G_{r-3,r+3} - G_{r,r} = (-1)^{r}(4p + 2q) \end{cases},$$

and, in general,

$$(16')\qquad \begin{cases} G_{r+s,r-s} - G_{r,r} = (-1)^{r+s+1}F_s(F_s p - F_{s+1}q) \\ G_{r-s,r+s} - G_{r,r} = (-1)^{r+s+1}F_s H_s \end{cases}.$$

From (16') we note that

$$(5')\qquad G_{r,n} = G_{n,r} + (-1)^n F_{r-n}\, q \quad .$$

We also note that

$$(17')\qquad \sum_{r=0}^{n-1} G_{r,r} = \begin{cases} 2 \cdot F_n H_n & \text{if } n \text{ even} \\ 2 \cdot F_{n+1} H_{n-1} - q & \text{if } n \text{ odd} \end{cases}.$$

Among the elementary properties that do not generalize are (10) and (12); however, the latter failure is the basis for the second generalization. The summation formulas in the introduction generalize as

$$(I')\qquad \sum_{n=0}^{r} G_{r-n,n} = \frac{2}{5}\left[(r + 1)L_{r+1} - F_{r+1}\right]p + \frac{1}{5}\left[2(r + 1)L_r + F_{r+1}\right]q \quad ,$$

$$(II')\qquad \sum_{n=0}^{r} (-1)^n G_{r-n,n} = qF_r \quad ,$$

$$(III')\qquad \sum_{n=0}^{r} \binom{r}{n} G_{r-n,n} = \frac{1}{5}\left[2^{r+1}L_{r+1} - 2\right]p + \frac{1}{5}\left[2^{r+1}L_r + 3\right]q \quad ,$$

(IV')
$$\sum_{n=0}^{r} (-1)^n \binom{r}{n} G_{r-n,n} = \begin{cases} q \text{ if } r = 0 \\ 5^{(r-1)/2} \cdot q \text{ if } r \text{ odd} \\ (-2p + q)5^{(r-2)/2} \text{ if } \frac{r}{2} \in J^+ \end{cases} .$$

II. SECOND GENERALIZATION

We define

(12")
$$H_{r,n} = H_r H_n + H_{r+n} .$$

It immediately follows that

(1")
$$H_{0,n} = H_n (q + 1) ,$$

(2")
$$H_{1,n} = pH_n + H_{n+1} ,$$

(3")
$$H_{r,n} = H_{r-1,n} + H_{r-2,n} ,$$

(4")
$$H_{r,n} = H_{r,n-1} + H_{r,n-2} ,$$

(5")
$$H_{r,n} = H_{n,r} .$$

See Table II for array H. We also note that

$$H_{r,n} = p^2 F_r F_n + q^2 F_{r-1} F_{n-1} + pq(F_r F_{n-1} + F_{r-1} F_n) + pF_{r+n} + qF_{r+n-1} .$$

We put

(6")
$$h_r(x) = \sum_{n=0}^{\infty} H_{r,n} x^n$$

and see that

(7")
$$h_0(x) = \frac{H_{0,0} + xH_{-1,0}}{1 - x - x^2} , \quad h_1(x) = \frac{H_{0,1} + xH_{-1,1}}{1 - x - x^2} .$$

Table II
Array H

r \ n	0	1	2	3	4
0	q^2 $+q$	pq $+p$	q^2 $+pq$ $+p$ $+q$	q^2 $+2pq$ $+2p$ $+q$	$2q^2$ $+3pq$ $+3p$ $+2q$
1	pq $+p$	p^2 $+p$ $+q$	p^2 $+pq$ $+2p$ $+q$	$2p^2$ $+pq$ $+3p$ $+2q$	$3p^2$ $+2pq$ $+5p$ $+3q$
2	q^2 $+pq$ $+p$ $+q$	p^2 $+pq$ $+2p$ $+q$	p^2 $+q^2$ $+2pq$ $+3p$ $+2q$	$2p^2$ $+q^2$ $+3pq$ $+5p$ $+3q$	$3p^2$ $+2q^2$ $+5pq$ $+8p$ $+5q$
3	q^2 $+2pq$ $+2p$ $+q$	$2p^2$ $+pq$ $+3p$ $+2q$	$2p^2$ $+q^2$ $+3pq$ $+5p$ $+3q$	$4p^2$ $+q^2$ $+4pq$ $+8p$ $+5q$	$6p^2$ $+2q^2$ $+7pq$ $+13p$ $+8q$
4	$2q^2$ $+3pq$ $+3p$ $+2q$	$3p^2$ $+2pq$ $+5p$ $+3q$	$3p^2$ $+2q^2$ $+5pq$ $+8p$ $+5q$	$6p^2$ $+2q^2$ $+7pq$ $+13p$ $+8q$	$9p^2$ $+4q^2$ $+12pq$ $+21p$ $+13q$

But by (3'') we have

(8'') $$h_r(x) = h_{r-1}(x) + h_{r-2}(x) ,$$

so that

(9'') $$h_r(x) = \frac{H_{0,r} + xH_{-1,r}}{1 - x - x^2} .$$

Putting

$$h(x,y) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} H_{r,n} x^r y^n$$

we have

$$h(x,y) = \sum_{r=0}^{\infty} \frac{H_{0,r} + yH_{-1,r}}{1 - y - y^2} x^r$$

(11")

$$= \frac{q(1 + q) + (p - q)(1 + q)(x + y) + xy(p^2 - p + 2q + 2pq + q^2)}{(1 - x - x^2)(1 - y - y^2)} .$$

From (12") we have

$$H_{r+s,r-s} - H_{r,r} = H_{r+s}H_{r-s} - H_r^2 ,$$

so that

(13") $$H_{r+1,r-1} - H_{r,r} = (-1)^r e ,$$

(14") $$H_{r+2,r-2} - H_{r,r} = (-1)^{r+1} e ,$$

(15") $$H_{r+3,r-3} - H_{r,r} = (-1)^r e 4 ,$$

(16") $$H_{r+s,r-s} - H_{r,r} = (-1)^{r+s+1} e F_s^2 ,$$

where $e = p^2 - pq - q^2$.

The summation formulas previously referred to generalize as

(I") $$\sum_{n=0}^{r} H_{r-n,n} = (r + 1)H_r + qH_r - \frac{e}{5} F_r + \frac{r}{5} [(H_{r+1} + H_{r-1})p + (H_r + H_{r-2})q] ,$$

(II") $$\sum_{n=0}^{r} (-1)^n H_{r-n,n} = \begin{cases} 0 \text{ if } r \text{ odd} \\ q(F_{r-1} + qF_{r+1} + 2pF_r) \\ + (p - p^2)F_r \text{ if } r \text{ even} \end{cases} ,$$

(III") $$\sum_{n=0}^{r} \binom{r}{n} H_{r-n,n} = 2^r H_r + \frac{1}{5} [2^r p(H_{r+1} + H_{r-1}) + 2^r q(H_r + H_{r-2}) - 2e] ,$$

(IV") $$\sum_{n=0}^{r} (-1)^n \binom{r}{n} H_{r-n,n} = \begin{cases} 0 \text{ if } r \text{ odd} \\ q + q^2 \text{ if } r = 0 \\ -2e\, 5^{(r-2)/2} \text{ if } r/2 \in J^+ \end{cases} .$$