DETERMINANTS AND IDENTITIES INVOLVING FIBONACCI SQUARES

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Determinants provide an unusual means of discovering identities involving elements of any Fibonacci sequence. In this paper, a determinant relationship believed to be new provides the derivation of several series of identities for Fibonacci sequences.

1. THE ALTERNATING LAMBDA NUMBER

First is displayed the theorem which provides the foundation for what follows. Only 3×3 determinants are discussed here, but the theorem is given in general.

Theorem. Let $A = (a_{ij})$ and $A^* = (a^*_{ij})$ be $n \times n$ matrices such that

$$a^*_{ij} = a_{ij} + (-1)^{i+j} k .$$

Then

$$\det A^* = \det A + k(\det C) ,$$

where $C = (c_{ij})$ is the $(n-1) \times (n-1)$ matrix given by

$$c_{ij} = a_{ij} + a_{i+1,j+1} + a_{i+1,j} + a_{i,j+1} .$$

Proof. Successively replace the k^{th} column by the sum of the $(k-1)^{st}$ and k^{th} columns for $k = n, n-1, \cdots, 2$. Then successively replace the k^{th} row by the sum of the $(k-1)^{st}$ and k^{th} row for $k = n, n-1, \cdots, 2$. The resulting determinant is

$$\begin{vmatrix} a_{11} + k & a_{11} + a_{12} & a_{12} + a_{13} & \cdots \\ a_{21} + a_{11} & c_{11} & c_{12} & \cdots \\ a_{31} + a_{21} & c_{21} & c_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \det A + k(\det C)$$

by noting that the determinant on the left can be expressed as the sum of two determinants by splitting the first column and then reversing the above steps for the determinant which does not contain k in the upper left corner.

Specifically, the theorem says that, for $n = 3$,

$$\begin{vmatrix} a+k & b-k & c+k \\ d-k & e+k & f-k \\ g+k & h-k & i+k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + k \begin{vmatrix} a+b+d+e & b+c+e+f \\ d+e+g+h & e+f+h+i \end{vmatrix}$$

<u>Definition.</u> We agree to call det C of the theorem the alternating lambda number of A, denoted by $\lambda_n(A)$.

The closely related lambda number of a matrix arising with the addition of a constant k to each element of a matrix has been discussed in [1], [2], and [3].

As an illustration of the theorem, evaluate $\det W_n$ for

$$W_n = \begin{bmatrix} L_n^2 & L_{n+1}^2 & L_{n+2}^2 \\ L_{n+1}^2 & L_{n+2}^2 & L_{n+3}^2 \\ L_{n+2}^2 & L_{n+3}^2 & L_{n+4}^2 \end{bmatrix}$$

where each element is the square of a Lucas number L_n, using the usual $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_n + L_{n+1}$. The value of the analogous $\det W_n^*$ where W_n^* is formed from W_n by replacing L_n by the Fibonacci number F_n, defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_n + F_{n+1},$$

has been given in [4] as $2(-1)^{n+1}$. It is not difficult to calculate $\lambda_a(W_n^*)$:

$$\lambda_a(W_n^*) = \begin{vmatrix} F_n^2 + F_{n+2}^2 + 2F_{n+1}^2 & F_{n+1}^2 + F_{n+3}^2 + 2F_{n+2}^2 \\ F_{n+1}^2 + F_{n+3}^2 + 2F_{n+2}^2 & F_{n+2}^2 + F_{n+4}^2 + 2F_{n+3}^2 \end{vmatrix} = \begin{vmatrix} L_{2n+2} & L_{2n+3} \\ L_{2n+3} & L_{2n+4} \end{vmatrix} = 5$$

Since

$$5F_n^2 = L_n^2 + (-1)^{n+1}4 ,$$

$$\det (5W_n^*) = \det W_n + (-1)^{n+1}4\cdot\lambda_a(5W_n^*)$$

$$5^3\cdot 2(-1)^{n+1} = \det W_n + (-1)^{n+1}4\cdot 5^2\cdot 5$$

$$\det W_n = (-1)^n 2\cdot 5^3 .$$

2. DETERMINANTS INVOLVING SQUARES OF ELEMENTS OF ANY FIBONACCI SEQUENCE

Consider the matrix

$$\text{(2.1)} \qquad A_n = \begin{bmatrix} H_n^2 & H_{n+1}^2 & H_{n+2}^2 \\ H_{n+1}^2 & H_{n+2}^2 & H_{n+3}^2 \\ H_{n+2}^2 & H_{n+3}^2 & H_{n+4}^2 \end{bmatrix}$$

where each element is the square of a member of a Fibonacci sequence $\{H_n\}$ defined by

$$H_1 = p, \quad H_2 = q, \quad H_{n+2} = H_{n+1} + H_n .$$

Since an identity for such Fibonacci sequences is

$$H_{n+3}^2 = 2H_{n+2}^2 + 2H_{n+1}^2 - H_n^2 ,$$

multiplying each element in columns two and three by (-2) and adding to column one yields the elements $-H_{n+3}^2$, $-H_{n+4}^2$, $-H_{n+5}^2$. Column exchanges show that

$$\det A_n = - \det A_{n+1} ,$$

so increasing the subscript by one in A_n only changes the sign of $\det A_n$, and $|\det A_n|$ is independent of n. It is not difficult (just messy) to evaluate $\det A_n$, then, by picking a value for n, calculating members of $\{H_n\}$ in terms of p and q, and using elementary algebra. This method of calculation for 3×3 determinants whose elements are squares of Fibonacci numbers was given by Fuchs and Erbacher in [4].

The results are

$$(2.2)\qquad \begin{aligned} \det A_n &= 2(-1)^n(q^2 - pq - p^2)^3 = 2(-1)^n D_H^3 \\ \lambda_a(A_n) &= 5(q^2 - pq - p^2)^2 = 5D_H^2 \end{aligned},$$

where $|D_H|$ is the characteristic number of the sequence (see [5]). If $\{H_n\} = \{F_n\}$, the Fibonacci sequence, $D_F = -1$ and $\det A_n = 2(-1)^{n+1}$.

The same method will allow the calculations of the values of several other determinants which follow.

$$(2.3)\qquad \det C_n = \begin{vmatrix} H_n^2 & H_{n+1}^2 & H_{n+2}^2 \\ H_{n+3}^2 & H_{n+4}^2 & H_{n+5}^2 \\ H_{n+6}^2 & H_{n+7}^2 & H_{n+8}^2 \end{vmatrix} = (-1)^n 64\, D_H^3 :$$

$$\lambda_n(C_n) = 160\, D_H^2$$

Continuing since also

$$H_{n+4}H_{n+2} = 2H_{n+3}H_{n+1} + 2H_{n+2}H_n - H_{n+1}H_{n-1},$$

we obtain (2.4) and (2.5):

$$(2.4)\qquad \det R_n = \begin{vmatrix} H_{n+1}H_{n-1} & H_{n+2}H_n & H_{n+3}H_{n+1} \\ H_{n+3}H_{n+1} & H_{n+4}H_{n+2} & H_{n+5}H_{n+3} \\ H_{n+4}H_{n+2} & H_{n+5}H_{n+3} & H_{n+6}H_{n+4} \end{vmatrix} = (-1)^{n+1} 3\, D_H^3 :$$

$$\lambda_a(R_n) = 5\, D_H^2 .$$

$$(2.5)\qquad \det S_n = \begin{vmatrix} H_{n+1}H_{n-1} & H_{n+2}H_n & H_{n+3}H_{n+1} \\ H_{n+4}H_{n+2} & H_{n+5}H_{n+3} & H_{n+6}H_{n+4} \\ H_{n+7}H_{n+5} & H_{n+8}H_{n+6} & H_{n+9}H_{n+7} \end{vmatrix} = (-1)^{n+1} 96\, D_H^3 :$$

$$\lambda_a(S_n) = 160\, D_H^2$$

Since

$$H_n^2 = H_{n+1} H_{n-1} + (-1)^n D_H ,$$

Equations (2.4) and (2.5) can be obtained in a second way with a minimum of effort by using the alternating lambda number theorem. For example, to find (2.5) using (2.3),

$$\det C_n = \det S_n + (-1)^n D_H \lambda_a(C_n)$$
$$64(-1)^n D_H^3 = \det S_n + (-1)^n D_H (160\, D_H^2)$$
$$\det S_n = (-1)^{n+1}\, 96\, D_H^3 \quad .$$

Also, notice that

$$\lambda_a(C_n) = \lambda_a(S_n) .$$

The identity

$$H_{n+6}^2 = 8H_{n+4}^2 - 8H_{n+2}^2 + H_n^2$$

allows one to use the method of Fuchs and Erbacher to find two more values:

$$\text{(2.6)} \qquad \det B_n = \begin{vmatrix} H_n^2 & H_{n+2}^2 & H_{n+4}^2 \\ H_{n+2}^2 & H_{n+4}^2 & H_{n+6}^2 \\ H_{n+4}^2 & H_{n+6}^2 & H_{n+8}^2 \end{vmatrix} = (-1)^n 18\, D_H^3 ;$$

$$\lambda_a(B_n) = 9\, D_H\, [(-1)^n 8H_{n+4}^2 + 13\, D_H]$$

$$\text{(2.7)} \qquad \begin{vmatrix} H_n^2 & H_{n+2}^2 & H_{n+4}^2 \\ H_{n+6}^2 & H_{n+8}^2 & H_{n+10}^2 \\ H_{n+12}^2 & H_{n+14}^2 & H_{n+16}^2 \end{vmatrix} = (-1)^n\, 2^{11}\, 3^3\, D_H^3 \quad .$$

Compare (2.6) with the Fibonacci result $(18)(-1)^{n+1}$ as given in [6], and notice that D_H^3 is a factor in each determinant value found in this section.

In (2.6) and (2.7) the alternating lambda numbers are not independent of n and hence are not useful in what follows. The alternating lambda number for (2.6) is interesting in that it depends upon the center element of B_n.

3. IDENTITIES FOR MEMBERS OF ANY FIBONACCI SEQUENCE $\{H_n\}$

Before we can continue, we must standardize our sequences. For purposes of forming a Fibonacci sequence, $H_1 = p$ and $H_2 = q$ are arbitrary integers. But surprisingly enough, if enough terms are written, each sequence has a subsequence of terms which alternate in sign as well as a subsequence in which all terms are of the same sign. Since we want a standard way of numbering the terms of these sequences in what follows, when we want the characteristic number

$$D_H = H_2^2 - H_2 H_1 - H_1^2$$

to be positive, then we take H_0 as the first member of the non-alternating subsequence, and H_1 as the second member. When we want $D_H < 0$, we take H_1 as the first or third member of the non-alternating subsequence. Note that $D_H = 5$ for $\{H_n\} = \{L_n\}$, and $D_H = -1$ for $\{H_n\} = \{F_n\}$. Now we are ready to develop several identities which relate two Fibonacci sequences.

The identity

$$L_n^2 + (-1)^{n+1} 4 = 5 F_n^2$$

suggests that we seek an identity relating two Fibonacci sequences $\{H_n\}$ and $\{G_n\}$. Returning to (2.1), form matrix A_n with elements from $\{H_n\}$ and matrix A_n^* with elements from $\{G_n\}$. If there exist two integers x and k such that

$$H_n^2 + (-1)^{n+1} x = k G_n^2 \quad ,$$

then the alternating lambda number theorem and (2.2) provide

$$\det A_n + (-1)^{n+1} x \lambda_a (kA_n^*) = \det (kA_n^*)$$

$$2(-1)^n D_H^3 + (-1)^{n+1} x(5k^2 D_G^2) = 2(-1)^n k^3 D_G^3$$

$$x = \frac{(D_H^3 - k^3 D_G^3)(2)}{5k^2 D_G^2}$$

If $-kD_G = D_H$, then $x = 4D_H/5$. Since x must be an integer, D_H must be a multiple of 5. A solution is given by $k = 5$, $D_H = 5(-D_G)$. Since 5 and multiples of 5 do occur as characteristic numbers, we have

(3.1) $$H_n^2 + (-1)^{n+1} \frac{4}{5} D_H = 5\, G_n^2 ,$$

where $\{H_n\}$ has the positive characteristic number D_H and $\{G_n\}$ has the negative characteristic number $D_G = -D_H/5$.

An example of a solution is given by the pairs of sequences

$$\{H_n\} = \{\cdots, 13, -6, 7, 1, 8, 9, \cdots\}$$

and

$$\{G_n\} = \{\cdots, 5, -1, 3, 2, 5, 7, \cdots\}$$

or their conjugates

$$\{H_n^*\} = \{\cdots, 8, -1, 7, 6, 13, \cdots\}$$

and

$$\{G_n^*\} = \{\cdots, 5, -2, 3, 1, 4, 5, \cdots\} .$$

Since $D_H = 55 > 0$, set $H_1 = 1$ and $H_1^* = 6$, but since $D_0 = -11 < 0$, take $G_1 = 3$ and $G_1^* = 4$. Using $\{H_n\}$ and $\{G_n\}$, notice that

(3.2) $$H_n^2 + (-1)^{n+1} 44 = 5\, G_n^2 .$$

Also note that

$$H_n + H_{n+2} = 5G_{n+1}$$

and

$$G_n + G_{n+2} = H_{n+1} .$$

Above, $\{H_n\}$ and $\{G_n\}$ were found by simply referring to a table of characteristic numbers. (See [5] and [7].) To write a pair of sequences $\{H_n\}$ and $\{G_n\}$ to satisfy (3.1), let $p > 0$ be an arbitrary integer. Let z be an integer such that

$$p \equiv 2z \pmod 5 .$$

Then $H_1 = p$ and $H_2 = z$ gives $D_H = 5m$ for some integer m, and

$$G_1 = \frac{2z - p}{5}, \qquad G_2 = \frac{2p + z}{5}$$

gives $\{G_n\}$ with $D_G = -m$. The justification is simple, for if $p \equiv 2z$ (mod 5), then

$$D_H = z^2 - pz - p^2 = (z - p)(z + p) - pz$$
$$\equiv (5k - z)(3z) - 2z^2 \equiv 15kz - 5z^2 \equiv 0 \pmod 5 .$$

The other statements follow by elementary algebra.

Solutions to (3.1) with $D_G = - D_H/5$ for $H_1 = 1, 2, \cdots, 7, \cdots, p, \cdots$ follow. In each case $u, t = 0, 1, 2, \cdots$.

Two more identities relating the two Fibonacci sequences $\{H_n\}$ and $\{G_n\}$ just described follow.

The identity

$$L_n L_{n+2} + (-1)^{n+1} = 5 F_{n-1}^2$$

D_H	$\{H_n\}$ (H_1, H_2)	$\{G_n\}$ (G_1, G_2)
$25t(t-1)+5$	$(1, -2+5t)$	$(2t-1, t)$
$25t^2-5$	$(2, 1+5t)$	$(2t, 1+t)$
$25t(t-1)-5$	$(3, -1+5t)$	$(2t-1, 1+t)$
$25t^2-20$	$(4, 2+5t)$	$(2t, 2+t)$
$25t(t-1)-25$	$(5, 5t)$	$(2t-1, 2+t)$
$25t^2-45$	$(6, 3+5t)$	$(2t, 3+t)$
$25t(t-1)-55$	$(7, 1+5t)$	$(2t-1, 3+t)$
...	...	...
$25t^2-5u^2$	$(2u, u+5t)$	$(2t, u+t)$
$25t(t-1)-5(u^2+u-1)$	$(2u+1, u+5t-2)$	$(2t-1, u+t)$

suggests searching for an identity of the form

$$H_n H_{n+2} + (-1)^{n+1} x = k G_{n+1}^2 .$$

The alternating lambda number theorem, (2.2) and (2.4) give

$$\det R_n + (-1)^{n+1} x \lambda_a (kA_n^*) = \det (kA_{n+1}^*)$$

$$3(-1)^{n+2} D_H^3 + (-1)^{n+1} x (5k^2 \cdot D_G^2) = 2(-1)^{n+1} k^3 D_G^3$$

$$x = \frac{2k^3 D_G^3 + 3 D_H^3}{5k^2 \cdot D_G^2}$$

If $kD_G = D_H$, then $x = D_H$, and we have the known identity

(3.3) $$H_n H_{n+2} + (-1)^{n+1} D_H = H_{n+1}^2 .$$

If $kD_G = -D_H$, then $x = D_H/5$. Again let $k = 5$ since D_H must be a multiple of 5, yielding

$$H_n H_{n+2} + (-1)^{n+1} D_H/5 = 5G_{n+1}^2 , \tag{3.4}$$

where the characteristic number of $\{G_n\}$ is $-D_H/5$.

A final derivation is suggested by the identity

$$L_n^2 + (-1)^n = 5F_{n+1}F_{n-1} .$$

Proceeding as before using (2.2) and (2.4),

$$H_n^2 + (-1)^n x = kG_{n+1}G_{n-1}$$

$$\det A_n + (-1)^n x\lambda_a(kR_n) = \det(kR_n)$$

$$2(-1)^n D_H^3 + (-1)^n x(5k^2D_G^2) = (-1)^{n+1} 3k^3 D_G^3$$

$$x = \frac{-3k^3D_G^3 - 2D_H^3}{5k^2D_G^2} .$$

If $D_H = -kD_G$, then $x = D_H/5$, and if $k = 5$, we have

$$H_n^2 + (-1)^n D_H/5 = 5G_{n+1}G_{n-1} , \tag{3.5}$$

where again $D_G = -D_H/5$. If $D_H = kD_G$, then $x = -D_H$, and taking $k = 1$ gives the known identity

$$H_n^2 + (-1)^{n+1} D_H = H_{n+1}H_{n-1} ,$$

which is the same as (3.3).

The possibilities are by no means exhausted by this paper.

REFERENCES

1. Marjorie Bicknell, "The Lambda Number of a Matrix: The Sum of its n^2 Cofactors," The American Mathematical Monthly, Vol. 72, No. 3, March, 1965, pp. 260-264.

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