ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

DEFINITIONS

The Fibonacci Numbers F_n and the Lucas Numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

PROBLEMS

B-226 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Find the smallest number in the Fibonacci sequence $1, 1, 2, 3, 5, \cdots$ that is not the sum of the squares of three integers.

B-227 Proposed by H. V. Krishna, Manipal Engineering College, Manipal, India.

Let $H_0, H_1, H_2, \cdots$ be a generalized Fibonacci sequence satisfying $H_{n+2} = H_{n+1} + H_n$ (and any initial conditions $H_0 = q$ and $H_1 = p$). Prove that

$$F_1H_3 + F_2H_6 + F_3H_9 + \cdots + F_nH_{3n} = F_nF_{n+1}H_{2n+1}.$$

B-228 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Extending the definition of the F_n to negative subscripts using

$$F_{-n} = (-1)^{n-1}F_n ,$$

prove that for all integers k, m, and n

$$(-1)^k F_n F_{m-k} + (-1)^m F_k F_{n-m} + (-1)^n F_m F_{k-n} = 0 .$$

B-229 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

Using the recursion formulas to extend the definition of F_n and L_n to all integers n, prove that for all integers k, m, and n

$$(-1)^k L_n F_{m-k} + (-1)^m L_k F_{n-m} + (-1)^n L_m F_{k-n} = 0 .$$

B-230 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let $\{C_n\}$ satisfy

$$C_{n+4} - 2C_{n+3} - C_{n+2} + 2C_{n+1} + C_n = 0$$

and let

$$G_n = C_{n+2} - C_{n+1} - C_n .$$

Prove that $\{G_n\}$ satisfies $G_{n+2} = G_{n+1} + G_n$.

B-231 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

A GFS (generalized Fibonacci sequence) $H_0, H_1, H_2, \cdots$ satisfies the same recursion formula

$$H_{n+2} = H_{n+1} + H_n$$

as the Fibonacci sequence but may have any initial values. It is known that

$$H_n H_{n+2} - H_{n+1}^2 = (-1)^n c ,$$

where the constant c is characteristic of the sequence. Let $\{H_n\}$ and $\{K_n\}$ be GFS and let

$$C_n = H_0K_n + H_1K_{n-1} + H_2K_{n-2} + \cdots + H_nK_0 \quad .$$

Show that

$$C_{n+2} = C_{n+1} + C_n + G_n \quad ,$$

where $\{G_n\}$ is a GFS whose characteristic is the product of those of $\{H_n\}$ and $\{K_n\}$.

SOLUTIONS

GENERALIZED FIBONACCI IDENTITY

B-208 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let

$$F_0 = 0, \ F_1 = 1, \ F_{n+2} = F_{n+1} + F_n, \ L_0 = 2, \ L_1 = 1, \ L_{n+2} = L_{n+1} + L_n.$$

Prove both of the following and generalize:

(a) $$F^2_{n+2} = 3F^2_{n+1} - F^3_n = 2(-1)^n$$

(b) $$L^2_{n+2} = 3L^2_{n+1} - L^2_n = 10(-1)^n \quad .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

In the paper by David Zeitlin, "Power Identities for Sequences Defined by $W_{n+2} = dW_{n+1} - cW_n$," this Quarterly, Vol. 3, No. 4, 1965, pp. 241-255, it is shown on page 251, Eq. (4.5) that

(1) $$H^2_{n+2} - 3H^2_{n+1} + H^2_n = 2(-1)^{n+1}(H_1^2 - H_1H_0 - H_0^2) \quad ,$$

where

$$H_{n+2} = H_{n+1} + H_n, \qquad n = 0, 1, \qquad .$$

Thus, (1) gives (a) for $H_n \equiv F_n$ and (b) for $H_n \equiv L_n$.

Also solved by Richard Blazej, Herta T. Freitag, Ralph Garfield, J. A. H. Hunter, C. B. A. Peck, A. G. Shannon, and the Proposer.

FURTHER GENERALIZATION

B-209 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California

Do the analogue of B-208 for the Pell sequence defined by

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+2} = 2P_{n+1} + P_n, \quad \text{and} \quad Q_n = P_n + P_{n-1} .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

In the paper quoted in B-208, there is given Eq. (3.1) on p. 245 which states that

$$(1) \qquad W_{n+2}^2 - (d^2 - 2c)W_{n+1}^2 + c^2W_n^2 = 2c^{n+1}(W_1^2 - dW_0W_1 + cW_0^2) ,$$

where

$$W_{n+2} = dW_{n+1} - cW_n .$$

Thus, for $d = 2$, $c = -1$, and $W_n \equiv P_n$, (1) gives

$$(2) \qquad P_{n+2}^2 - 6P_{n+1}^2 + P_n^2 = 2(-1)^{n+1} .$$

Since

$$Q_{n+2} = 2Q_{n+1} + Q_n ,$$

we obtain from (1) for $d = 2$, $c = -1$, and $W_n \equiv Q_n$, $Q_0 = 1$, $Q_1 = 1$,

$$(3) \qquad Q_{n+2}^2 - 6Q_{n+1}^2 + Q_n^2 = 4(-1)^n .$$

Also solved by Herta T. Freitag, Ralph Garfield, A. G. Shannon, and the Proposer.

SUMMING OF FIBONACCI RECIPROCALS

B-210 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove that $S > 803/240$, where

$$S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \cdots .$$

Solution by Peter A. Lindstrom, Genesee Community College, Batavia, New York.

Consider the finite sum S_n, where

$$S_n = (1/F_1) + (1/F_2) + \cdots + (1/F_n) .$$

Then one finds that

$$240\,S_{13} = 240 + 240 + 120 + 80 + 48 + 30 + 18\frac{6}{13} + 11\frac{9}{21} + 7\frac{2}{34}$$
$$+ 4\frac{20}{55} + 2\frac{62}{89} + 1\frac{96}{144} + 1\frac{7}{233} .$$

and hence $240\,S_{13} > 803$. Then $S > S_{13} > 803/240$.

Also solved by R. Garfield, C. B. A. Peck, and the Proposer.

FIBONACCI WITH A GEOMETRIC PROGRESSION

B-211 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California. (Corrected)

Let F_n be the n^{th} term in the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$. Solve the recurrence

$$D_{n+1} = 2D_n + F_{2n+1}$$

subject to the initial condition $D_1 = 1$.

Composite of solutions by Herta T. Freitag, Hollins, Virginia, and R. Garfield, College of Insurance, New York, New York.

The condition $D_2 = 3$ is unnecessary and is indeed false since the recurrence gives $D_2 = 2D_1 + F_3 = 2 \cdot 1 + 2 = 4$.

By writing a few terms in the D_n sequence it is easy to show that

$$D_{n+1} = 2^n D_1 + 2^{n-1}F_3 + 2^{n-2}F_5 + \cdots + 2F_{2n-1} + F_{2n+1}\,.$$

Using the Binet formula and summing geometric progressions, we find that

$$D_n = F_{2n+2} - 2^n\,.$$

It is easier to prove this by mathematical induction than to check the details.

Also solved by the Proposer.

A QUESTION WITH MANY ANSWERS

B-212 Proposed by Tomas Djerverson, Albrook College, Tigertown on the Rio.

Give examples of interesting functions f and g such that

$$f(m,n) = g(m + n) - g(m) - g(n)\,.$$

(One example is $f(m,n) = mn$ and

$$g(n) = \binom{n}{2} = n(n - 1)/2\,.\,)$$

EPS Editor's Note. We tabulate some of the submitted answers as follows:

Solver	f(m, n)	g(m)
Proposer	mn	$\binom{m}{2} = m(m - 1)/2$
Herta T. Freitag	mn	$m(m + c)/2$, c constant
Herta T. Freitag	$g(m)g(n)$	$r^m - 1$, r constant
John W. Milsom	$2mn$	m^2
John W. Milsom	$3mn(m + n)$	m^3
Phil Mana	$\log\binom{m+n}{m}$	$\log(m!)$

UNFRIENDLY SUBSETS ON A LINE OR CIRCLE

B-213 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Given n points on a straight line, find the number of subsets (including the empty set) of the n points in which consecutive points are not allowed. Also find the corresponding number when the points are on a circle.

Solution by Theodore J. Cullen, Cal Poly, Pomona, California.

Let T_n be the solution for the line. It is easily seen that $F_1 = 2$ and $T_2 = 3$. For $n \geq 3$, let p be an extreme point, i.e., p has only one neighbor. Then the subsets can be divided into two types, those with p absent and those with p present. Clearly there are T_{n-1} of the first type and T_{n-2} of the second type, so that

$$T_n = T_{n-1} + T_{n-2} \, .$$

Therefore $T_n = F_{n+2}$ for $n \geq 1$, where $F_1 = F_2 = 1$ and

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 3$, the Fibonacci numbers.

Let V_n be the solution for the circle. One can check that $V_1 = 2$, $V_2 = 3$, $V_3 = 4$. For $n \geq 4$ let p be any fixed point, and again consider subsets with p absent and then p present. The numbers of these are T_{n-1} and T_{n-3}, respectively, so that

$$V_n = T_{n-1} + T_{n-3} = F_{n+1} + F_{n-1} = L_n \, ,$$

the n^{th} Lucas number.

Also solved by Sister Marion Beiter, Herta T. Freitag, and the Proposer.