SOME PROPERTIES OF THIRD-ORDER RECURRENCE RELATIONS

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1. INTRODUCTION

In this paper, we set out to establish some results about third-order recurrence relations, using a variety of techniques.

Consider a third-order recurrence relation

(1.1) $$S_n = PS_{n-1} + QS_{n-2} + RS_{n-3} \quad (n \geq 4), \quad S_0 = 0,$$

where P, Q, and R are arbitrary integers.

Suppose we get the sequence

(1.2) $$\{J_n\}, \quad \text{when} \quad S_1 = 0, \; S_2 = 1, \; \text{and} \; S_3 = P,$$

and the sequence

(1.3) $$\{K_n\}, \quad \text{when} \quad S_1 = 1, \; S_2 = 0, \; \text{and} \; S_3 = Q,$$

and the sequence

(1.4) $$\{L_n\}, \quad \text{when} \quad S_1 = 0, \; S_2 = 0, \; \text{and} \; S_3 = R.$$

It follows that

$$K_1 = J_2 - J_1, \quad K_2 = J_3 - PJ_2,$$

and for $n \geq 3$,

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(1.5) $$K_n = QJ_{n-1} + RJ_{n-2} ,$$

and

(1.6) $$L_n = RJ_{n-1} .$$

These sequences are generalizations of those discussed by Feinberg [2], [3] and Waddill and Sacks [6].

2. GENERAL TERMS

If the auxiliary equation

$$x^3 - Px^2 - Qx - R = 0$$

has three distinct real roots, suppose that they are given by α, β, γ.

According to the general theory of recurrence relations, J_n can be represented by

(2.1) $$J_n = A\alpha^{n-1} + B\beta^{n-1} + C\gamma^{n-1} ,$$

where

$$A = \frac{\alpha}{(\beta - \alpha)(\gamma - \alpha)}, \quad B = \frac{\beta}{(\gamma - \beta)(\alpha - \beta)} ,$$

and

$$C = \frac{\gamma}{(\alpha - \gamma)(\beta - \gamma)}$$

(A, B and C are determined by J_1, J_2, and J_3.)

The first few terms of $\{J_n\}$ are

$$(J_1) = 0,\ 1,\ P,\ P^2 + Q,\ P^3 + 2PQ + R,\ P^4 + 3P^2Q + 2PR + Q^2 .$$

These terms can be determined by the use of the formula

$$J_{n+2} = \sum_{i=0}^{[n/3]} R^i \sum_{j=0}^{[n/2]} a_{nij} P^{n-3i-2j} Q^j , \tag{2.2}$$

where a_{nij} satisfies the partial difference equation

$$a_{nij} = a_{n-1,i,j} + a_{n-2,i,j-1} + a_{n-3,i-1,j} \tag{2.3}$$

with initial conditions

$$a_{noj} = \binom{n-j}{j}$$

and

$$a_{nio} = \binom{n-2i}{i} .$$

For example,

$$\begin{aligned} J_5 &= a_{300} P^3 + a_{301} PQ + a_{310} R \\ &= P^3 + 2PQ + R . \end{aligned}$$

Formula (2.2) can be proved by induction. In outline, the proof uses the basic recurrence relation (1.1) and then the partial difference equation (2.3). The result follows because

$$PJ_{n+1} = \sum_{i=0}^{[(n-1)/3]} R^i \sum_{j=0}^{[(n-1)/2]} a_{n-1,i,j} P^{n-3i-2j} Q^j ,$$

$$QJ_n = \sum_{i=0}^{[(n-2)/3]} R^i \sum_{j=1}^{[n/2]} a_{n-2,i,j-1} P^{n-3i-2j} Q^j ,$$

$$RJ_{n-1} = \sum_{i=1}^{[n/3]} R^i \sum_{j=0}^{[(n-3)/2]} a_{n-3,i-1,j} P^{n-3i-2j} Q^j .$$

By using the techniques developed for second-order recurrence relations, it can be shown that

$$(2.4) \quad (P + Q + R - 1)\sum_{r=1}^{n} J_r = J_{n+3} + (1 - P)J_{n+2} + (1 - P - Q)J_{n+1} - 1.$$

It can also be readily confirmed that the generating function for $\{J_n\}$ is

$$(2.5) \quad \sum_{n=0}^{\infty} J_n x^n = x^2(1 - Px - Qx^2 - Rx^3)^{-1} .$$

3. THE OPERATOR E

We define an operator E, such that

$$(3.1) \quad EJ_n = J_{n+1} ,$$

and suppose, as before, that there exist 3 distinct real roots, α, β, γ of the auxiliary equation

$$x^3 - Px^2 - Qx - R = 0 .$$

This can be written as

$$(x - \alpha)(x - \beta)(x - \gamma) = (x^2 - px + q)(x - \gamma) = 0 ,$$

where

$$p = \alpha + \beta = P - \gamma ,$$

and $q = \alpha\beta$.

The recurrence relation

$$J_n = PJ_{n-1} + QJ_{n-2} + RJ_{n-3}$$

can then be expressed as

$$(E^3 - PE^2 - QE - R)J_n = 0 \qquad \text{(replacing } n \text{ by } n + 3)$$

or

$$(3.2) \qquad (E^2 - pE + q)(E - \gamma)J_n = 0 ,$$

which becomes

$$(3.3) \qquad (E^2 - pE + q)u_n = 0$$

or

$$u_{n+2} - pu_{n+1} + qu_n = 0$$

if we let

$$(E - \gamma)J_n = u_n ,$$

where $\{u_n\}$ is defined by

$$(3.4) \qquad u_{n+2} = pu_{n+1} - qu_n, \quad (n \ge 0), \quad u_0 = 0, \quad u_1 = 1 .$$

In other words,

$$(3.5) \qquad u_n = J_{n+1} - \gamma J_n$$

and the extensive properties developed for $\{u_n\}$ can be utilized for $\{J_n\}$.

In particular,

$$u_n^2 - u_{n-1} \cdot u_{n+1} = q^{n-1} \tag{3.6}$$

becomes

$$(J_{n+1} - \gamma J_n)^2 - (J_n - \gamma J_{n-1})(J_{n+2} - \gamma J_{n+1}) = q^{n-1} .$$

This gives us

$$(J_{n+1}^2 - J_n J_{n+2}) + \gamma(J_{n+1}J_n - J_{n+2}J_{n-1}) + \gamma^2(J_n^2 - J_{n+1}J_{n-1}) = q^{n-1} . \tag{3.7}$$

Another identity for $\{J_n\}$ analogous to (3.6) is developed below as (4.4).

Since

$$\begin{aligned} J_n &= u_{n-1} + \gamma J_{n-1} \\ &= u_{n-1} + \gamma(u_{n-2} + J_{n-2}) \\ &= u_{n-1} + \gamma u_{n-2} + \gamma^2(u_{n-3} + J_{n-3}) \end{aligned}$$

then

$$J_n = \sum_{r=1}^{n} \gamma^{n-r} u_{r-1} , \tag{3.8}$$

which may be a more useful form of the general term than those expressed in (2.1) and (2.2).

4. USE OF MATRICES

Matrices can be used to develop some of the properties of these sequences. In general, we have

$$\begin{bmatrix} S_5 \\ S_4 \\ S_3 \end{bmatrix} = \begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} S_4 \\ S_3 \\ S_2 \end{bmatrix} = \begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 \begin{bmatrix} S_3 \\ S_2 \\ S_1 \end{bmatrix}$$

and so, by finite induction,

$$\begin{bmatrix} S_n \\ S_{n-1} \\ S_{n-2} \end{bmatrix} = \begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} S_3 \\ S_2 \\ S_1 \end{bmatrix} \tag{4.1}$$

Again, since

$$\begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} P^2 + Q & PQ + R & PR \\ P & Q & R \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} J_4 & K_4 & RJ_3 \\ J_3 & K_3 & RJ_2 \\ J_2 & K_2 & RJ_1 \end{bmatrix}$$

we can show by induction that

$$\underset{\sim}{S}^n = \begin{bmatrix} P & Q & R \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} J_{n+2} & K_{n+2} & RJ_{n+1} \\ J_{n+1} & K_{n+1} & RJ_n \\ J_n & K_n & RJ_{n-1} \end{bmatrix} \tag{4.2}$$

The corresponding determinants give

$$(\det \underset{\sim}{S})^n = R^n = \begin{vmatrix} J_{n+2} & K_{n+2} & RJ_{n+1} \\ J_{n+1} & K_{n+1} & RJ_n \\ J_n & K_n & RJ_{n-1} \end{vmatrix} \tag{4.3}$$

By the repeated use of (1.5), we can show that

$$\begin{vmatrix} J_{n+2} & K_{n+2} & RJ_{n+1} \\ J_{n+1} & K_{n+1} & RJ_n \\ J_n & K_n & RJ_{n-1} \end{vmatrix} = R^2 \begin{vmatrix} J_{n+1} & J_n & J_{n+1} \\ J_{n+1} & J_{n-1} & J_n \\ J_n & J_{n-2} & J_{n-1} \end{vmatrix}$$

and

$$\begin{vmatrix} J_{n+2} & J_n & J_{n+1} \\ J_{n+1} & J_{n-1} & J_n \\ J_n & J_{n-2} & J_{n-1} \end{vmatrix} = R^{n-2} \tag{4.4}$$

which is analogous to

$$u_n^2 - u_{n-1} \cdot u_{n+1} = q^{n-1} \tag{4.5}$$

for the second-order sequence $\{u_n\}$ defined above, (3.4). In the more general case, we get

$$\underset{\sim}{S}_n = \begin{bmatrix} S_{n+3} & S_{n+1} & S_{n+2} \\ S_{n+2} & S_n & S_{n+1} \\ S_{n+1} & S_{n-1} & S_n \end{bmatrix} = \underset{\sim}{S}^{n-1} \underset{\sim}{S}_1$$

and the corresponding determinants are

$$\begin{vmatrix} S_{n+3} & S_{n+1} & S_{n+2} \\ S_{n+2} & S_n & S_{n+4} \\ S_{n+1} & S_{n-1} & S_n \end{vmatrix} = R^{n-1} \begin{vmatrix} S_4 & S_2 & S_3 \\ S_3 & S_1 & S_2 \\ S_2 & S_0 & S_1 \end{vmatrix} .$$

Matrices can also be used to develop expressions for

$$\sum_{n=0}^{\infty} \frac{J_n}{n!}, \quad \sum_{n=0}^{\infty} \frac{K_n}{n!}, \quad \sum_{n=0}^{\infty} \frac{L_n}{n!} ,$$

by adapting and extending a technique used by Barakat [1] for the Lucas polynomials.

Let

$$\underset{\sim}{X} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with a trace

$$P = a_{11} + a_{22} + a_{33}, \qquad \det \underset{\sim}{X} = R ,$$

and

$$Q = \sum_{i,j=1}^{3} a_{ij}a_{ji} - a_{ii}a_{jj} , \qquad (i \neq j) .$$

For example,

$$\underset{\sim}{X} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

satisfies the conditions.

The characteristic equation of $\underset{\sim}{X}$ is

$$\lambda^3 - P\lambda^2 - Q\lambda - R = 0$$

and so, by the Cayley-Hamilton Theorem [4],

$$\underset{\sim}{X}^3 = P\underset{\sim}{X}^2 + Q\underset{\sim}{X} + R\underset{\sim}{I} .$$

Thus

$$\underset{\sim}{X}^4 = P\underset{\sim}{X}^3 + Q\underset{\sim}{X}^2 + R\underset{\sim}{X}$$

$$= (P^2 + Q)\underset{\sim}{X}^2 + (PQ + R)\underset{\sim}{X} + PR\underset{\sim}{I}$$

and so on, until

(4.6) $$\underset{\sim}{X}^n = J_n\underset{\sim}{X}^2 + K_n\underset{\sim}{X} + L_n\underset{\sim}{I} .$$

Now, the exponential of a matrix $\underset{\sim}{X}$ of order 3 is defined by the infinite series

(4.7) $$e^{\underset{\sim}{X}} = \underset{\sim}{I} + \frac{1}{1!}\underset{\sim}{X} + \frac{1}{2!}\underset{\sim}{X} + \cdots ,$$

where $\underset{\sim}{I}$ is the unit matrix of order 3.

Substitution of (4.6) into (4.7) yields

(4.8) $$e^{\underset{\sim}{X}} = \underset{\sim}{X}^2\sum_{n=0}^{\infty}\frac{J_n}{n!} + \underset{\sim}{X}\sum_{n=0}^{\infty}\frac{K_n}{n!} + \underset{\sim}{I}\sum_{n=0}^{\infty}\frac{L_n}{n!} .$$

Sylvester's matrix interpolation formula [5] gives us

(4.9) $$e^{\underset{\sim}{X}} = \sum_{\lambda_1,\lambda_2,\lambda_3} e^{\lambda_1}\frac{(\underset{\sim}{X} - \lambda_2\underset{\sim}{I})(\underset{\sim}{X} - \lambda_3\underset{\sim}{I})}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} ,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\underset{\sim}{X}$.

Simplification of (4.9) yields

(4.10) $$e^{\underset{\sim}{X}} = \frac{\sum_{\lambda_1,\lambda_2,\lambda_3}\{e^{\lambda_1}(\lambda_3 - \lambda_2)\underset{\sim}{X}^2 + e^{\lambda_1}(\lambda_3^2 - \lambda_2^2)\underset{\sim}{X} + e^{\lambda_1}\lambda_2\lambda_3(\lambda_3 - \lambda_2)\underset{\sim}{I}\}}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}$$

By comparing coefficients of $\underset{\sim}{X}^n$ in (4.8) and (4.10), we get

$$\sum_{n=0}^{\infty} \frac{J_n}{n!} = \frac{\sum_{\lambda_1,\lambda_2,\lambda_3} e^{\lambda_1}(\lambda_3 - \lambda_2)}{\prod_{\lambda_1,\lambda_2,\lambda_3} (\lambda_1 - \lambda_2)} ,$$

$$\sum_{n=0}^{\infty} \frac{K_n}{n!} = \frac{\sum e^{\lambda_1}(\lambda_3^2 - \lambda_2^2)}{\prod(\lambda_1 - \lambda_2)} ,$$

$$\sum_{n=0}^{\infty} \frac{L_n}{n!} = \frac{\sum e^{\lambda_1}\lambda_2\lambda_3(\lambda_3 - \lambda_2)}{\prod(\lambda_1 - \lambda_2)} .$$

The authors hope to develop many other properties of third-order recurrence relations.

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