ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-192 Proposed by Ronald Alter, University of Kentucky, Lexington, Kentucky.

If

$$c_n = \sum_{j=0}^{3n+1} \binom{6n+3}{2j+1}(-11)^j ,$$

prove that

$$c_n = 2^{6n+3}N , \qquad (N \text{ odd}, \ n \geq 0) .$$

H-193 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona.

Prove or disprove: If

$$x + y + z = 2^{2n+1} - 1 \qquad \text{and} \qquad x^3 + y^3 + z^3 = 2^{6n+1} - 1,$$

then $6n + 1$ and $2^{6n+1} - 1$ are primes.

H-194 Proposed by H. V. Krishna, Manipal Engineering College, Manipal, India.

Solve the Diophantine equations:

(i) $$x^2 + y^2 \pm 5 = 3xy$$

(ii) $$x^2 + y^2 \pm e = 3xy\,,$$

where

$$e = p^2 - pq - q^2\,,$$

p, q positive integers.

SOLUTIONS

BINET GAINS IDENTITY

H-180 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{k=0}^{n} \binom{n}{k}^3 F_k = \sum_{2k \le n} \frac{(n+k)!}{(k!)^3 (n-2k)!} F_{(2n-3k)}$$

$$\sum_{k=0}^{n} \binom{n}{k}^3 L_k = \sum_{2k \le n} \frac{(n+k)!}{(k!)^3 (n-2k)!} L_{(2n-3k)}\,,$$

where F_k and L_k denote the k^{th} Fibonacci and Lucas numbers, respectively.

Solution by David Zeitlin, Minneapolis, Minnesota.

A more general result is that

(1) $$\sum_{k=0}^{n} \binom{n}{k}^3 b^{n-k} a^k W_k = \sum_{2k \le n} \frac{(n+k)!}{(k!)^3 (n-2k)!} b^k a^k W_{2n-3k}\,,$$

where $W_{n+2} = aW_{n+1} + bW_n$, $n = 0, 1, \cdots$. For $a = b = 1$, we obtain the desired results with $W_k = F_k$ and $W_k = L_k$.

Proof. From a well-known result*, we note that

$$(2)\qquad \sum_{k=0}^{n}\binom{n}{k}^3 x^k = \sum_{2k\le n}\frac{(n+k)!}{(k!)^3(n-2k)!}\,x^k(x+1)^{n-2k}\,.$$

Set $x = (ay)/b$ in (2) to obtain:

$$(3)\qquad \sum_{k=0}^{n}\binom{n}{k}^3 b^{n-k}a^k y^k = \sum_{2k\le n}\frac{(n+k)!}{(k!)^3(n-2k)!}\,b^k a^k y^k(ay+b)^{n-2k}\,.$$

Let α,β be the roots of $y^2 = ay + b$. Noting that $W_n = C_1\alpha^n + C_2\beta^n$, we obtain (1) by addition of (3) for $y = \alpha$ and $y = \beta$.

Remarks. If $a = 2x$, $b = -1$, then with $W_k = T_k(x)$, the Chebyshev polynomial of the first kind, we obtain from (1)

$$(4)\qquad \sum_{k=0}^{n}\binom{n}{k}^3(-1)^{n-k}(2x)^k T_k(x) = \sum_{2k\le n}\frac{(n+k)!}{(k!)^3(n-2k)!}\,(-2x)^k T_{2n-3k}(x)\,.$$

For $a = 2$, $b = 1$, one may choose $W_k = P_k$, the Pell sequence.

Let $V_0 = 2$, $V_1 = a$, and $V_{k+2} = aV_{k+1} + bV_k$. Then, from (1), we obtain the general result

$$(5)\qquad \begin{aligned}\sum_{k=0}^{n}\binom{n}{k}^3 V_k^k\left\{(-1)^{m+1}b^m\right\}^{n-k} W_{mk+p}\\ = \sum_{2k\le n}\frac{(n+k)!}{(k!)^3(n-2k)!}\left((-1)^{m+1}b^m V_m\right)^k W_{m(2n-3k)+p}\end{aligned}\,,$$

for $m,p = 0, 1, \cdots$.

It should be noted that (1) is valid for equal roots, i.e., $\alpha = \beta$.

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*J. Riordan, Combinatorial Identities, p. 41.

Also solved by F. D. Parker, A. G. Shannon, and the Proposer.

SUM-ER TIME

H-181 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Prove the identity

$$\sum_{m,n=0}^{\infty} (am + cn)^m (bm + dn)^n \frac{u^m v^n}{m!\,n!} = \frac{1}{(1 - ax)(1 - dy) - bcxy},$$

where

$$u = xe^{-(ax+by)}, \qquad v = ye^{-(cx+dy)}.$$

Solution by the Proposer.

$$\sum_{m,n=0}^{\infty} (am + cn)^m (bm + dn)^n \frac{u^m v^n}{m!\,n!}$$

$$= \sum_{m,n=0}^{\infty} (am + cn)^m (bm + dn)^n \frac{x^m y^n}{m!\,n!} e^{-(am+cn)x-(bm+dn)y}$$

$$= \sum_{m,n=0}^{\infty} (am + cn)^m (bm + dn)^n \frac{x^m y^n}{m!\,n!} \sum_{j=0}^{\infty} (-1)^j \frac{(am+cn)^j}{j!} x^j \sum_{k=0}^{\infty} (-1)^k \frac{(bm+dn)^k}{k!} y$$

$$= \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m!\,n!} \sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{j+k} \binom{m}{j} \binom{n}{k} \left[a(m-j) + c(n-k)\right]^m \left[b(m-j) + d(n-k)\right]^n$$

$$= \sum_{m,n=0}^{\infty} \frac{x^m y^n}{m!\,n!} \sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{m+n-j-k} \binom{m}{j} \binom{n}{k} (aj + ck)^m (bj + dk)^n .$$

But

$$S_{m,n} = \sum_{j=0}^{m}\sum_{k=0}^{n} (-1)^{m+n-j-k}\binom{m}{j}\binom{n}{k}(aj + ck)^m(bj + dk)^n$$

$$(*) \qquad = \sum_{r=0}^{m}\sum_{n=0}^{n}\binom{m}{r}\binom{n}{a}a^{m-r}o^r b^{n-s}d^s \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^{m+n-r-s}$$

$$\cdot \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}d^{r+s} \quad .$$

Since

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}j^t = \begin{cases} 0 & (t < m) \\ m! & (t = m) \end{cases} ,$$

we need only consider those terms in (*) such that

$$\begin{cases} m + n - r - s \geq m \\ \qquad\qquad r + s \geq n \end{cases} ,$$

that is, $r + s = n$.

We therefore get

$$S_{m,n} = m!\,n! \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r}a^{m-r}d^{n-r}(bc)^r ,$$

so that

$$\sum_{m,n=0}^{\infty} (am + cn)^m (bm + dn)^n \frac{u^m v^n}{m!\,n!}$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{\min(m,n)} \binom{m}{r}\binom{n}{r} a^{m-r} d^{n-r} (bc)^r$$

$$= \sum_{r=0}^{\infty} (bcxy)^r \sum_{m,n=r}^{\infty} \binom{m+r}{r}\binom{n+r}{r} (ax)^m (dy)^n$$

$$= \sum_{r=0}^{\infty} (bcxy)^r (1 - ax)^{-r-1} (1 - dy)^{-r-1}$$

$$= (1 - ax)^{-1}(1 - dy)^{-1}\left\{1 - \frac{bcxy}{(1 - ax)(1 - dy)}\right\}^{-1}$$

$$= \left\{(1 - ax)(1 - dy) - bcxy\right\}^{-1}$$

ARRAY OF HOPE

H-183 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California.

Consider the display indicated below.

1						
1	1					
2	2	1				
5	4	3	1			
13	9	7	4	1		
34	22	16	11	5	1	
89	56	38	27	16	6	1
...						

Pascal Rule of Formation Except for Prescribed Left Edge.

(i) Find an expression for the row sums.

(ii) Find a generating function for the row sums.

(iii) Find a generating function for the rising diagonal sums.

Solution by the Proposer.

i) An inspection of the array reveals that the row sums are F_{2n+1} $(n = 0, 1, 2, \cdots)$

ii) If the columns are multiplied by $1, 2, 3, \cdots$ sequentially to the right, then the row sums have the generating function,

$$\frac{(1 - x)}{(1 - 3x + x^2)} \cdot \frac{(1 - x)}{(1 - 2x)} .$$

Thus the row sums are the convolution of the two sequences:

a) $A_1 = 1, \quad A_n = F_{2n+1} \quad (n \geq 1)$ and

b) $B_1 = 1, \quad B_n = 2^{n-1} \quad (n \geq 1)$.

iii) The rising diagonal sums, E_n, are the convolution of the two sequences:

c) $C_n = F_{n-1}$ and

d) $D_n = F_{2n-1} \quad (n = 0, 1, 2, \cdots)$.

Hence

$$\frac{(1 - x)^3}{(1 - x - x^2)(1 - 3x + x^2)} = \sum_{n=0}^{\infty} E_n x^n .$$

FIBO-CYCLE

H-184 Proposed by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Define the cycle α_n $(n = 1, 2, \cdots)$ as follows:

i) $$\alpha_n = (1, 2, 3, 4, \cdots, F_n) ,$$

where F_n denotes the n^{th} Fibonacci number. Now construct a sequence of permutations

$$\left\{ \alpha_n^{F_i} \right\}_{i=1}^{\infty} , \qquad (n = 1, 2, \cdots)$$

(ii) $$\alpha_n^{F_{i+2}} = \alpha_n^{F_i} \cdot x_n^{F_{i+1}} \qquad (i \geq 1) \quad .$$

Finally, define a sequence

$$\{u_n\}_{n=1}^{\infty}$$

as follows: u_n is the period of

$$\left\{\alpha_n^{F_i}\right\}_{i=1}^{\infty}$$

i.e., u_n is the smallest positive integer such that

(iii) $$\alpha_n^{F_{i+u_n}} = \alpha_n^{F_i} \qquad (i \geq N) .$$

a) Find a closed form expression for u_n.
b) If possible, show $N = 1$ is the minimum positive integer for which iii) holds for all $n = 1, 2, \cdots$.

Solution by the Proposer.

Since α_n is of order F_n, it follows that the exponents of α_n may be replaced by residues mod F_n and u_n is thus the period of the Fibonacci sequence mod F_n. Therefore $u_1 = u_2 = 1$, $u_3 = 3$. Consider the First n residue classes of the Fibonacci seuqence, mod F_n $(n \geq 4)$; $1, 1, 2, 3, \cdots, F_{n-1}, 0$. The $(n+1)^{st}$ residue class is $F_{n-1} = 1 + (F_{n-1} - 1)$ and $(2n-1)^{st}$ class is

$$F_{n-1} + F_{n-1}(F_{n-1} - 1) = F_{n-1}^2 .$$

However,

$$F_{n+1} = F_n + F_{n-1} \qquad (n \geq 2)$$

and

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad (n \geq 2)$$

implies

$$F_{n-1}^2 \equiv (-1)^n \pmod{F_n} .$$

If n is even $(n \geq 4)$, we have $F_{n-1}^2 \equiv 1 \pmod{F_n}$ and $u_n = 2n$. If n is odd $(n > 4)$, $F_{n-1}^2 \equiv -1 \pmod{F_n}$ and $u_n = 4n$.

From the above, it is obvious that $N = 1$ is the smallest positive integer for which (iii) holds for all $n = 1, 2, \cdots$. It is interesting to note that

$$\{u_n | n = 1, 2, \cdots\} \cap \{F_n | n = 1, 2, \cdots\} = \{F_1, F_4, F_6, F_9, F_{12}, \cdots\} .$$

[Continued from page 282.]

NOTE ON SOME SUMMATION FORMULAS

by

$$\frac{\displaystyle\prod_{i=1}^{s_0+s_1+s_2+\cdots} (k + s_1 + 2s_2 + 3s_3 + \cdots + i)}{s_0!s_1!s_2!\cdots} .$$

REFERENCE

1. L. Carlitz, "Some Summation Formulas," Fibonacci Quarterly, Vol. 9 (1971), pp. 28-34.