FIBONACCI NUMBERS AND GEOMETRY

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The Fibonacci relations we are going to develop represent a special case of algebra. If we are able to relate them to geometry we should take a quick look at the way algebra and geometry can be tied together.

One use of geometry is to serve as an illustration of an algebraic relation. Thus

$$(a + b)^2 = a^2 + 2ab + b^2$$

is exemplified by Figure 1.

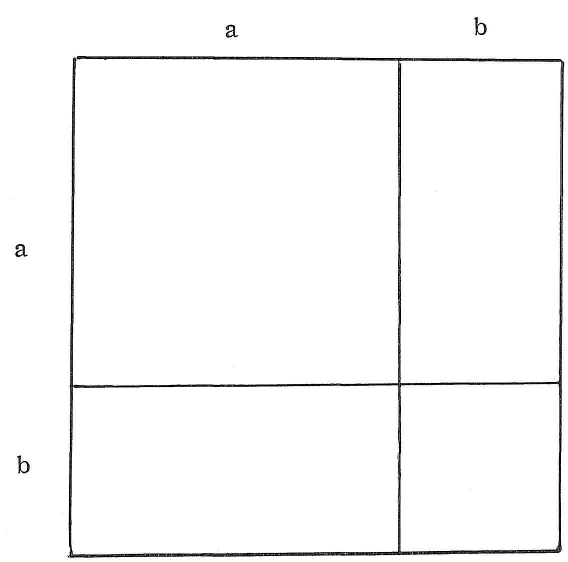


Figure 1

A second use of geometry is to provide a PROOF of an algebraic relation. As we ordinarily conceive the Pythagorean Theorem (though this was not the original thought of the Greeks) we tend to think of it as an algebraic relation on the sides of the triangle, namely,

$$c^2 = a^2 + b^2 \ .$$

One proof by geometry of this algebraic relation is shown in Figure 2.

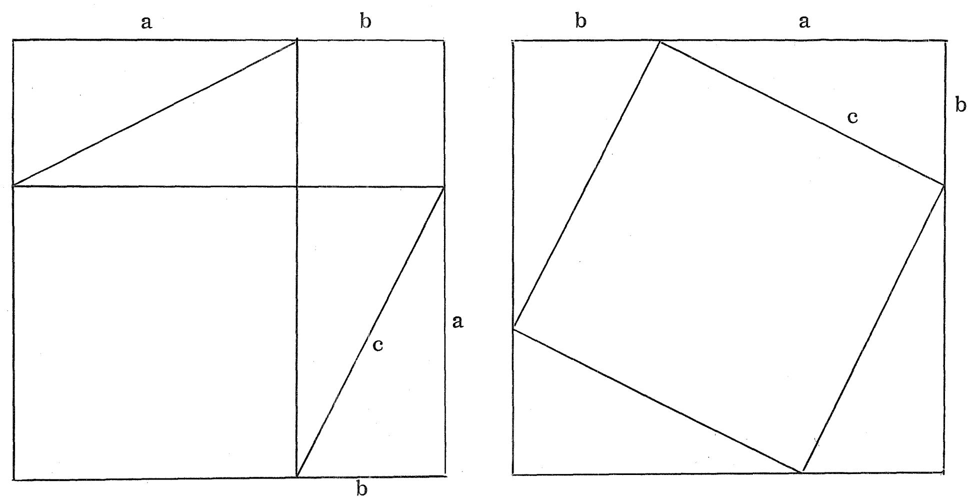


Figure 2

In summary, geometric figures may illustrate algebraic relations or they may serve as proofs of these relations. In our development, the main emphasis will be on proof though obviously illustration occurs simultaneously as well.

SUM OF FIBONACCI SQUARES

In the standard treatment of the Fibonacci sequence, geometry enters mainly at one point: summing the squares of the first n Fibonacci numbers. Algebraically, it can be shown by intuition and proved by induction that the sum of the squares of the first n Fibonacci numbers is

$$F_n F_{n+1} \ .$$

But there is a geometric pattern which ILLUSTRATES this fact beautifully as shown in Figure 3.

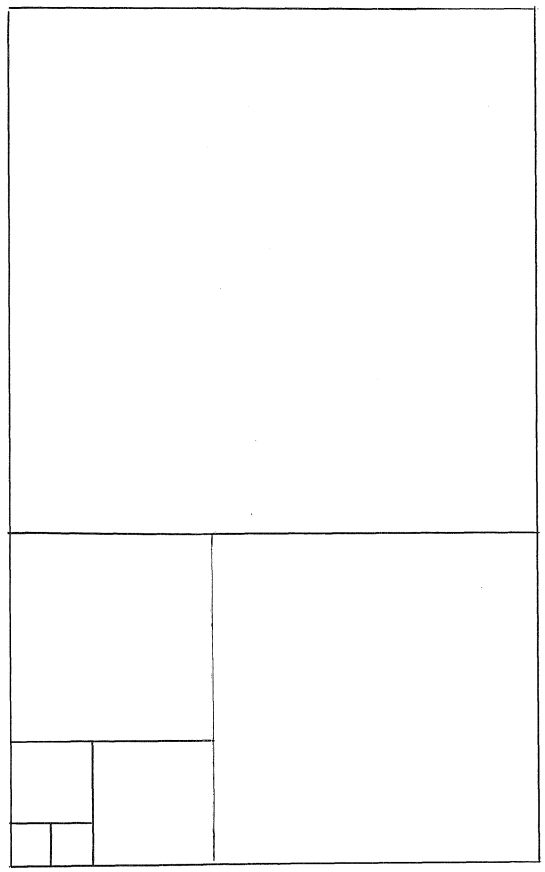
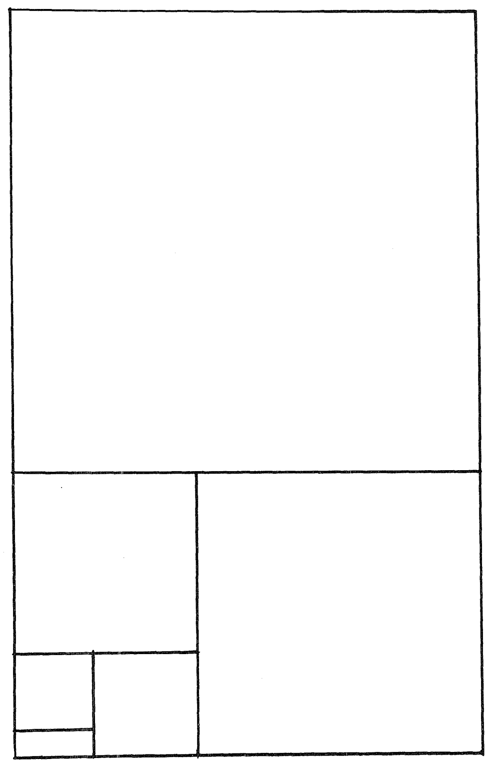


Figure 3

The figure is built up as follows. We put down two unit squares which are the squares of F_1 and F_2. Now we have a rectangle of dimensions 1 by 2. On top of this can be placed a square of side 2 (F_3) which gives a 2 by 3 rectangle. Then to the right can be set a square of side 3 (F_4) which produces a rectangle of sides 3 by 5. On top of this can be placed a square of side 5 (F_5) which gives a 5 by 8 (F_5F_6) rectangle, and so on.

This is where geometry begins and ends in the usual treatment of Fibonacci sequences. For if one tries to produce a similar pattern for the sum of the squares of any other Fibonacci sequence, there is an impasse. To meet this road block the following detour was conceived.

Suppose we are trying to find the sum of the squares of the first n Lucas numbers. Instead of starting with a square, we put down a rectangle whose sides are 1 and 3, the first and second Lucas numbers. (Figure 4 illustrates the general procedure.) Then on the side of length 3 it is possible to place



$$\sum_{k=1}^{n} T_k^2 = T_n T_{n+1} - T_1 (T_2 - T_1)$$

Figure 4

a square of side 3: this gives a 3 by 4 rectangle. Against this can be set a square of side 4 thus producing a 4 by 7 rectangle. On this a square of side 7 is laid giving a 7 by 11 rectangle. Thus the same process that operated for the Fibonacci numbers is now operating for the Lucas numbers. The only difference is that we began with a 1 by 3 rectangle instead of a 1 by 1 square. Hence, if we subtract 2 from the sum we should have the sum of the squares of the first n Lucas numbers. The formula for this sum is thus:

$$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2 . \tag{5}$$

Using a direct geometric approach it has been possible to arrive at this algebraic formula with a minimum of effort. By way of comparison it may be noted that the intuitional algebraic route usually leads to difficulties for students.

Still more striking is the fact that by using the same type of procedure it is possible to determine the sum of the squares of the first n terms of ANY Fibonacci sequence. We start again by drawing a rectangle of sides T_1 and T_2 (see Fig. 4). On the side T_2 we place a square of side T_2 to give a rectangle of sides T_2 and T_3. Against the T_3 side we set a square of side T_3 to produce a rectangle of sides T_3 and T_4. The operation used in the Fibonacci and Lucas sequences is evidently working again in this general case, the sum being $T_n T_{n+1}$ if we end with the n^{th} term squared. But instead of having the squre of T_1 as the first term, we used instead $T_1 T_2$. Thus it is necessary to subtract

$$T_1 T_2 - T_1^2$$

from the sum to arrive at the sum of the squares of the first n terms of the sequence. The formula that results is:

$$\sum_{k=1}^{n} T_k^2 = T_n T_{n+1} - T_1(T_2 - T_1) = T_n T_{n+1} - T_1 T_0 . \tag{6}$$

ILLUSTRATIVE FORMULAS

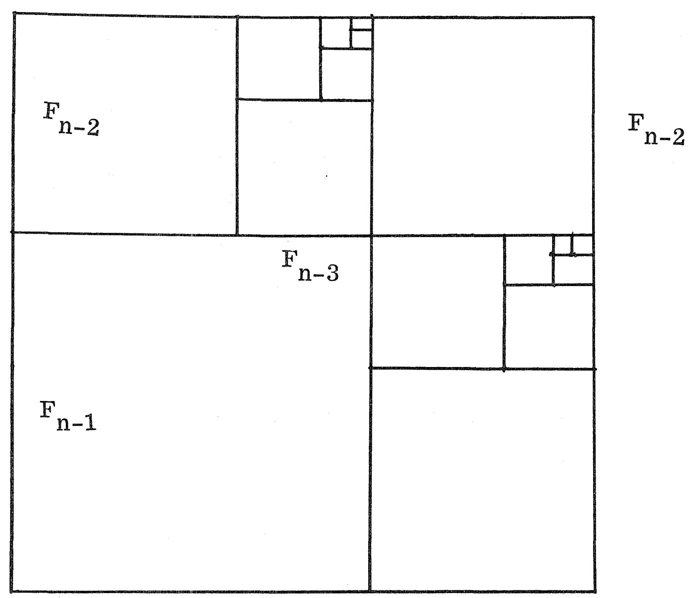
The design in Figure 1 for $(a + b)^2 = a^2 + 2ab + b^2$ can be used to illustrate Fibonacci relations that result from this algebraic identity. For example, Formulas (2), (3), and (4) could be employed for this purpose. Thus

$$L_n^2 = F_{n+1}^2 + 2 F_{n+1} F_{n-1} + F_{n-1}^2 .$$

This evidently leads to nothing new but the algebraic relations can be exemplified in this way as special cases of a general algebraic relation which is depicted by geometry.

LARGE SQUARE IN ONE CORNER

We shall deal with a number of geometric patterns which can be employed in a variety of ways in many cases. In the first type we place in one corner of a given figure the largest possible Fibonacci (or Lucas) square that will fit into it. Take, for example, a square whose side is F_n. (See Fig. 5.)



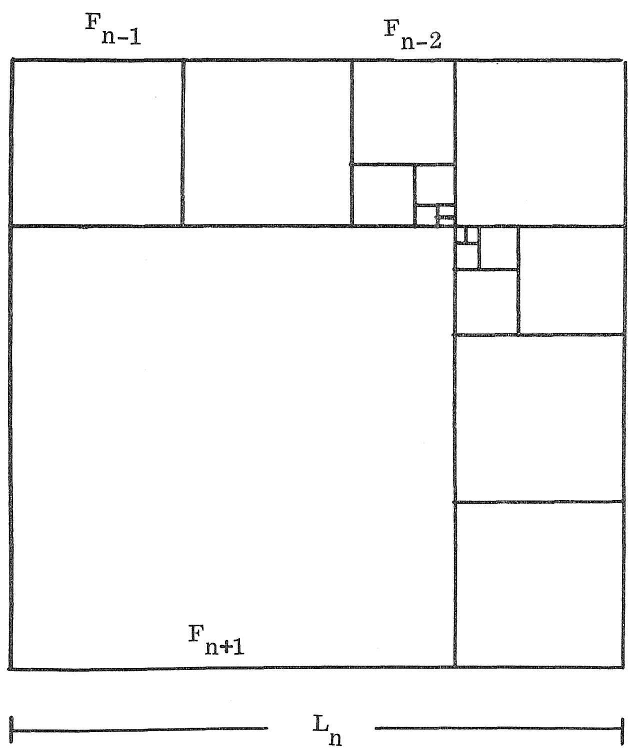
$$F_n^2 = F_{n-1}^2 + 3F_{n-2}^2 + 2\sum_{k=1}^{n-3} F_k^2$$

Figure 5

This being the sum of F_{n-1} and F_{n-2}, a square of side F_{n-1} can be put into one corner and its sides extended. In the opposite corner is a square of side F_{n-2}. From the two rectangles can be taken squares of side F_{n-2} leaving two smaller rectangles of dimensions F_{n-2} and F_{n-3}. But by what was found in the early part of this discussion, such a rectangle can be represented as the sum of the first $n - 3$ Fibonacci squares. We thus arrive at the formula:

$$F_n^2 = F_{n-1}^2 + 3F_{n-2}^2 + 2\sum_{k=1}^{n-3} F_k^2 \quad . \tag{7}$$

As a second example, take a square of side $L_n = F_{n+1} + F_{n-1}$. (See Fig. 6.)



$$L_n^2 = F_{n+1}^2 + 5F_{n-1}^2 + 2\sum_{k=1}^{n-2} F_k^2$$

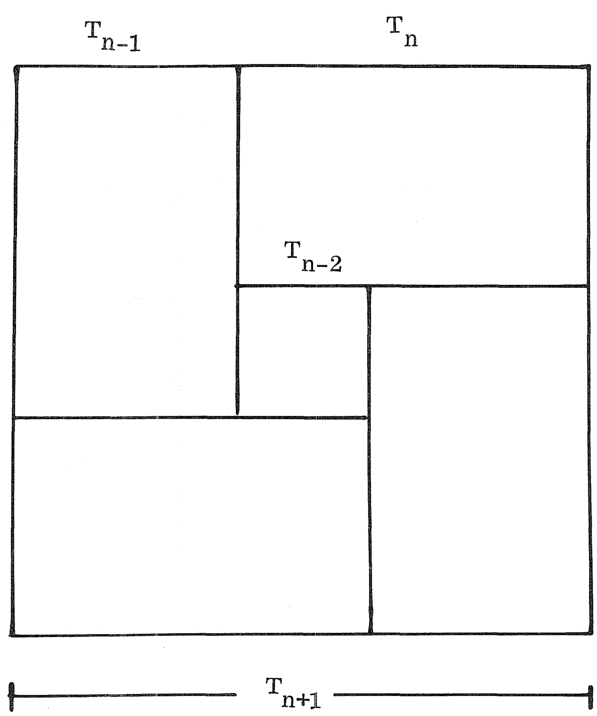
Figure 6

In one corner is a square of side F_{n+1} and in the opposite a square of side F_{n-1}. The rectangles have dimensions F_{n+1} and F_{n-1}. But F_{n+1} equals $2F_{n-1} + F_{n-2}$ by (3), so that each rectangle contains two squares of side F_{n-1} and a rectangle of sides F_{n-1} and F_{n-2}. Thus the following formula results:

$$L_n^2 = F_{n+1}^2 + 5F_{n-1}^2 + 2\sum_{k=1}^{n-2} F_k^2 . \tag{8}$$

CYCLIC RECTANGLES

A second type of design leading to Fibonacci relations is one that may be called cyclic rectangles. Take a square of side T_{n+1}, a general Fibonacci number. Put in one corner a rectangle of sides T_n and T_{n-1} (Fig. 7). The



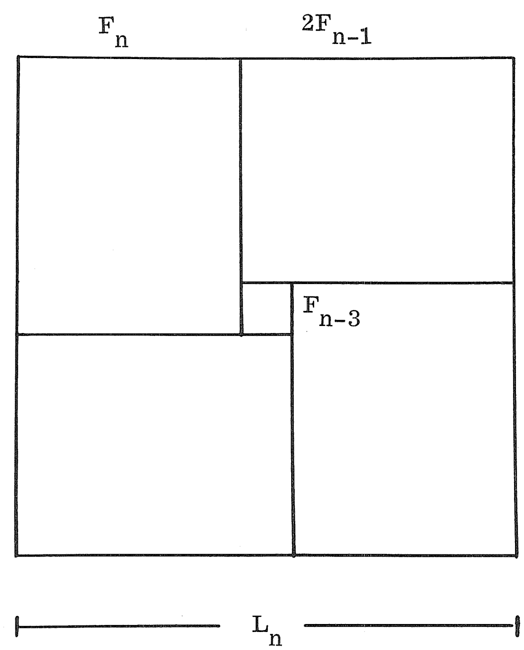
$$T_{n+1}^2 = 4T_nT_{n-1} + T_{n-2}^2$$

Figure 7

process can be continued until there are four such rectangles in a sort of whorl with a square in the center. This square has side $T_n - T_{n-1}$ or T_{n-2}. Accordingly the general relation for all Fibonacci sequences results:

$$T_{n+1}^2 = 4T_nT_{n-1} + T_{n-2}^2. \tag{9}$$

As another example of this type of configuration consider a square of side L_n and put in each corner a rectangle of dimensions $2F_{n-1}$ by F_n. (See Fig. 8.)



$$L_n^2 = 8F_nF_{n-1} + F_{n-3}^2$$

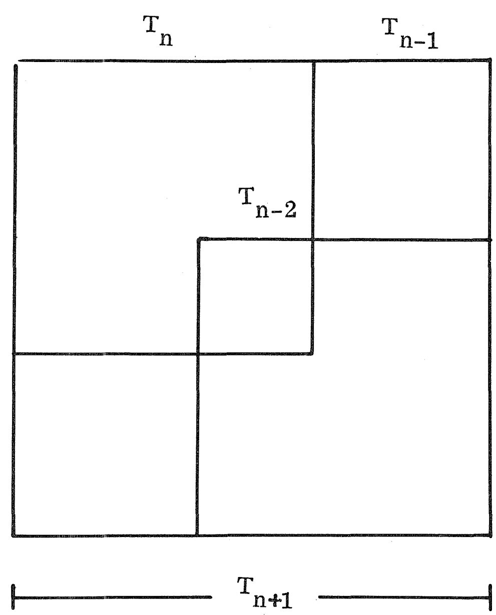
Figure 8

Again, there is a square in the center with side $2F_{n-1} - F_n$ or $F_{n-1} - F_{n-2} = F_{n-3}$. Hence:

$$L_n^2 = 8F_nF_{n-1} + F_{n-3}^2 \; . \tag{10}$$

OVERLAPPING SQUARES IN TWO OPPOSITE CORNERS

Construct a square whose side is T_{n+1} which equals $T_n + T_{n-1}$. In two opposite corners place squares of side T_n (Fig. 9). Since T_n is greater



$$T_{n+1}^2 = 2T_n^2 + 2T_{n-1}^2 - T_{n-2}^2$$

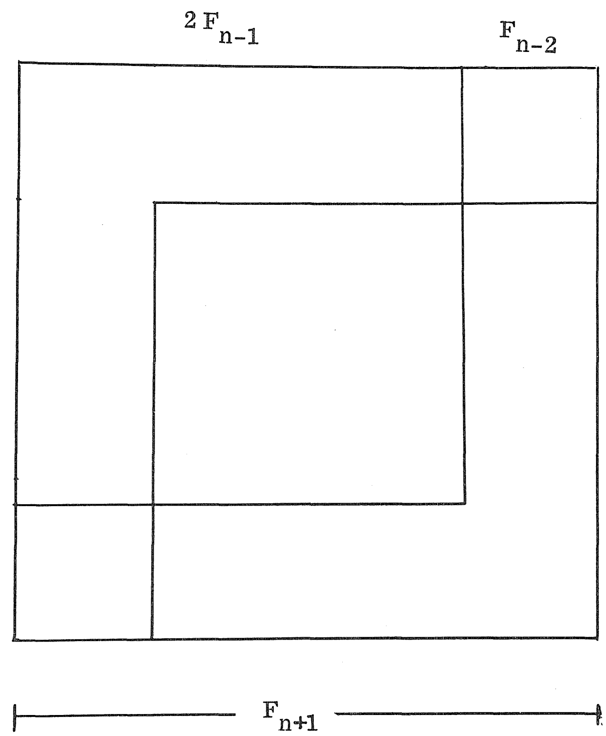
Figure 9

than half of T_{n+1} it follows that these squares must overlap in a square. The side of this square is $T_n - T_{n-1} = T_{n-2}$. The entire square is composed of two squares of side T_n and two squares of side T_{n-1}. But since the area of the central square of side T_{n-2} has been counted twice, it must be subtracted once to give the proper result. Thus:

$$T_{n+1}^2 = 2T_n^2 + 2T_{n-1}^2 - T_{n-2}^2 , \tag{11}$$

a result applying to all Fibonacci sequences.

Example 2. Take a square of side $F_{n+1} = 2F_{n-1} + F_{n-2}$. In opposite corners, place squares of side $2F_{n-1}$. (See Fig. 10). Then the overlap square



$$F_{n+1}^2 = 8F_{n-1}^2 + 2F_{n-2}^2 - L_{n-2}^2$$

Figure 10

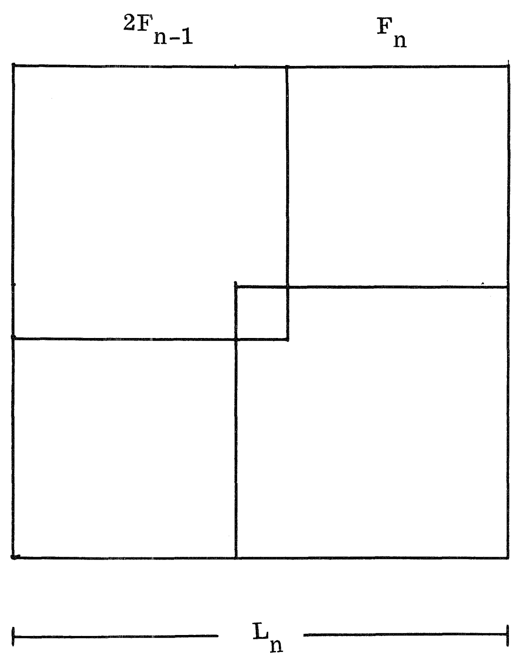
in the center has side $2F_{n-1} - F_{n-2} = F_{n-1} + F_{n-3} = L_{n-2}$. Thus:

(12) $$F_{n+1}^2 = 8F_{n-1}^2 + 2F_{n-2}^2 - L_{n-2}^2 \quad .$$

Third example. A square of side $L_n = F_{n+1} + F_{n-1}$ has a central overlapping square of side $F_{n+1} - F_{n-1} = F_n$. Accordingly:

(13) $$L_n^2 = 2F_{n+1}^2 + 2F_{n-1}^2 - F_n^2 \quad .$$

Final example. In a square of side $L_n = 2F_{n-1} + F_n$, place in two opposite corners squares of side $2F_{n-1}$. The overlap square in the center has side $2F_{n-1} - F_n = F_{n-1} - F_{n-2} = F_{n-3}$. (See Fig. 11.)



$$L_n^2 = 8F_{n-1}^2 + 2F_n^2 - F_{n-3}^2$$

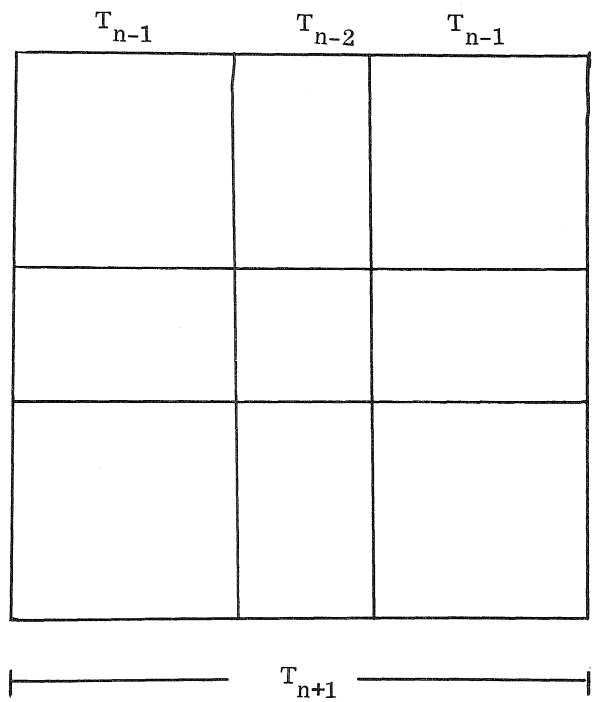
Figure 11

Hence:

(14) $$L_n^2 = 8F_{n-1}^2 + 2F_n^2 - F_{n-3}^2 .$$

NON-OVERLAPPING SQUARES IN FOUR CORNERS

Consider the relation $T_{n+1} = 2T_{n-1} + T_{n-2}$. Each side of the square can be divided into segments T_{n-1}, T_{n-2}, T_{n-1} in that order (Fig. 12).



$$T^2_{n+1} = 4T^2_{n-1} + 4T_{n-1}T_{n-2} + T^2_{n-2}$$

Figure 12

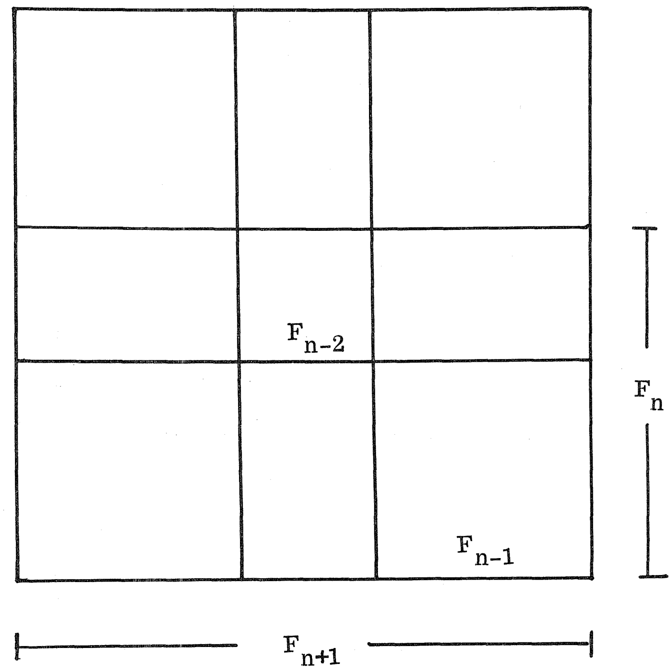
There are now four squares of side T_{n-1} in the corners, a square of side T_{n-2} in the center and four rectangles of dimensions T_{n-1} and T_{n-2}. From

(15) $$T^2_{n+1} = 4T^2_{n-1} + 4T_{n-1}T_{n-2} + T^2_{n-2}$$

which applies to ALL Fibonacci sequences.

OVERLAPPING SQUARES IN FOUR CORNERS

We start with $F_{n+1} = F_n + F_{n-1}$ and put four squares of side F_n in the corners (Fig. 13). Clearly there is a great deal of overlapping. The



$$F^2_{n+1} = 4F^2_n - 4F_{n-1}F_{n-2} - 3F^2_{n-2}$$

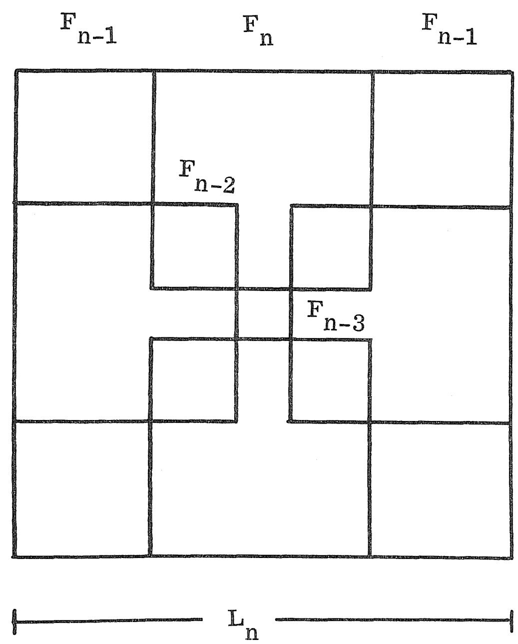
Figure 13

square at the center of side $F_n - F_{n-1} = F_{n-2}$ is covered four times; the four rectangles are each found in two of the corner squares so that this rectangle must be subtracted out four times. The central square being covered four times must be subtracted out three times. As a result the following formula is obtained:

$$F^2_{n+1} = 4F^2_n - 4F_{n-1}F_{n-2} - 3F^2_{n-2} \tag{16}$$

OVERLAPPING SQUARES PROJECTING FROM THE SIDES

We start with the relation $L_n = F_n + 2F_{n-1}$ and divide the side into segments F_{n-1}, F_n, F_{n-1} in that order (Fig. 14). On the F_n segments build squares which evidently overlap as shown. The overlap squares in the corners of these four squares have a side $F_n - F_{n-1} = F_{n-2}$ while the central



$$L_n^2 = 4F_n^2 + 4F_{n-1}^2 - 4F_{n-2}^2 + F_{n-3}^2$$

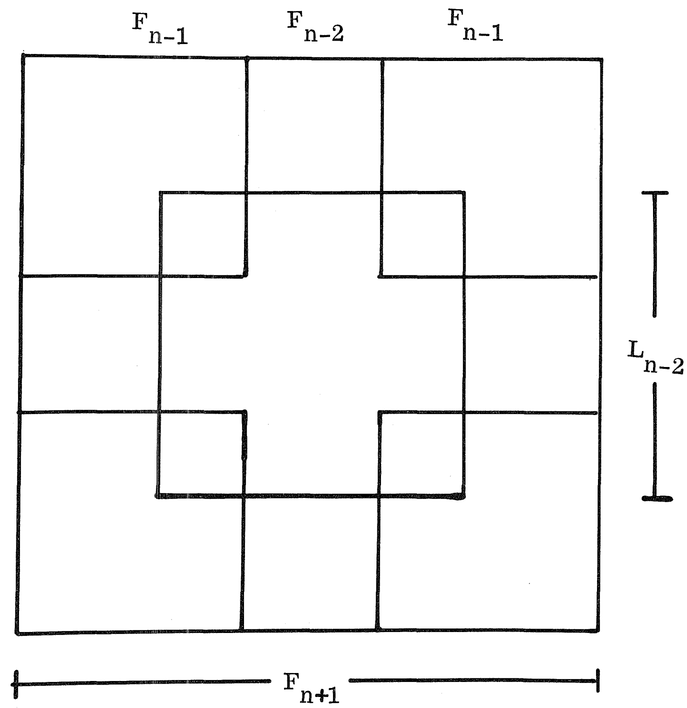
Figure 14

square has a side $L_n - 2F_n = F_{n+1} + F_{n-1} - 2F_n = 2F_{n-1} - F_n = F_{n-1} - F_{n-2} = F_{n-3}$. Taking the overlapping areas into account gives the relation:

$$L_n^2 = 4F_n^2 + 4F_{n-1}^2 - 4F_{n-2}^2 + F_{n-3}^2 \; . \tag{17}$$

FOUR CORNER SQUARES AND A CENTRAL SQUARE

A square of side $F_{n+1} = 2F_{n-1} + F_{n-2}$ has its sides divided into segments F_{n-1}, F_{n-2}, F_{n-1} in that order (Fig. 15). In each corner, a square of side F_{n-1} is constructed. Then a centrally located square of side L_{n-2} is constructed. It may be wondered where the idea for doing this came from. Since



$$F^2_{n+1} = 4F^2_{n-1} + 4F^2_{n-2} + L^2_{n-2} - 4F^2_{n-3}$$

Figure 15

$$L_{n-2} = F_{n-1} + F_{n-3} = F_{n-2} + 2F_{n-3},$$

it follows that such a square would project into the corner squares in the amount of F_{n-3}, thus giving three squares of this dimension. Taking overlap into account leads to the formula:

$$F^2_{n+1} = 4F^2_{n-1} + 4F^2_{n-2} + L^2_{n-2} - 4F^2_{n-3} \tag{18}$$

CONCLUSION

In this all too brief session we have explored some of the relations of Fibonacci numbers and geometry. It is clear that there is a field for developing

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