NOTE ON SOME SUMMATION FORMULAS

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In a recent paper [1], the writer has proved the following multiple summation formula:

$$(1)\qquad \sum_{s_1, s_2, \cdots, =0}^{\infty} \frac{(k + 2s_1 + 3s_2 + \cdots)!}{s_1!\, s_2! \cdots (k + s_1 + 2s_2 + \cdots)!} \cdot \frac{u_1^{s_1} u_2^{s_2} \cdots}{(1 + u_1 + u_2 + u_3 + \cdots)^{2s_1 + 3s_2 + \cdots}} = \frac{(1 + u_1 + u_2 + \cdots)^{k+1}}{1 - u_1 - 2u_2 - 3u_3 - \cdots} \qquad (k = 0, 1, 2, \cdots),$$

where the series

$$(2)\qquad u_1 + u_2 + u_3 + \cdots ,$$

is absolutely convergent but otherwise arbitrary.

In the present note we should like to point out that (1) admits of the following extension:

$$(3)\qquad \sum_{s_0, s_1, s_2, \cdots =0}^{\infty} \frac{(k + s_0 + 2s_1 + 3s_2 + \cdots)!}{s_0!\, s_1!\, s_2! \cdots (k + s_1 + 2s_2 + \cdots)!} \cdot \frac{u_0^{s_0} u_1^{s_1} u_2^{s_2} \cdots}{(1 + u_0 + u_1 + u_2 + \cdots)^{s_0 + 2s_1 + \cdots}}$$

$$= \frac{(1 + u_0 + u_1 + u_2 + \cdots)^{k+1}}{1 - u_1 - 2u_2 - 3u_3 - \cdots} \qquad (k = 0, 1, 2, \cdots),$$

where again the series (2) is absolutely convergent.

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Proof of (3). By (1),

$$\frac{(1+u_0+u_1+u_2+\cdots)^{k+1}}{1-u_1-2u_2-3u_3-\cdots} = \frac{(1+u_1+u_2+\cdots)^{k+1}}{1-u_1-2u_2-3u_3-\cdots}\left(\frac{1+u_0+u_1+u_2+\cdots}{1+u_1+u_2+\cdots}\right)^{k+2}$$

$$= \sum_{s_1,s_2,\cdots,=0}^{\infty} \frac{(k+2s_1+3s_2+\cdots)!}{s_1!\,s_2!\cdots(k+s_1+2s_2+\cdots)!} \frac{u_1^{s_1}u_2^{s_2}\cdots}{(1+u_1+u_2+\cdots)^{2s_1+3s_2+\cdots}}$$

$$\cdot \left(\frac{1+u_0+u_1+\cdots}{1+u_1+u_2+\cdots}\right)^{k+1}$$

$$= \sum_{s_1,s_2,\cdots,=0}^{\infty} \frac{(k+2s_1+3s_2+\cdots)!}{s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!} \frac{u_1^{s_1}u_2^{s_2}\cdots}{(1+u_0+u_1+\cdots)^{2s_1+3s_2+\cdots}}$$

$$\cdot \left(1-\frac{u_0}{1+u_0+u_1+\cdots}\right)^{-k-2s_1-3s_2-\cdots-1}$$

$$= \sum_{s_1,s_2,\cdots,=0}^{\infty} \frac{(k+2s_1+3s_2+\cdots)!}{s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!} \frac{u_1^{s_1}u_2^{s_2}}{(1+u_0+u_1+\cdots)^{2s_1+3s_2+\cdots}}$$

$$\cdot \sum_{s_0=0}^{\infty} \binom{k+s_0+2s_1+3s_2+\cdots}{s_0} \frac{u_0^{s_0}}{(1+u_0+u_1+\cdots)^{s_0}}$$

$$= \sum_{s_0,s_1,s_2,\cdots=0}^{\infty} \frac{(k+s_0+2s_1+3s_2+\cdots)!}{s_0!s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!} \frac{u_0^{s_0}u_1^{s_1}u_2^{s_2}\cdots}{(1+u_0+u_1+\cdots)^{s_0+2s_1+3s_2+\cdots}}.$$

This evidently proves (3).

Exactly as in [1], we can show that (3) holds for arbitrary k, provided we replace the coefficient

$$\frac{(k+s_0+2s_1+3s_2+\cdots)!}{s_0!s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!}$$

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