GENERALIZED FIBONACCI NUMBERS IN PASCAL'S PYRAMID

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1. INTRODUCTION

It is well known that the Fibonacci numbers are the rising diagonals of Pascal's triangles. Harris and Styles [2] generalized the Fibonacci numbers to other diagonals. Hoggatt and Bicknell further generalized these to other Pascal triangles in [3]. Mueller in [5] discusses sums taken over planar sections of Pascal's pyramid. Here we further extend the results in [5] to many relations with the Fibonacci numbers.

In [1] many nice derivations were obtained using generating functions for the columns of Pascal's binomial triangle. Further results will be forthcoming in [6]. The earliest results were laid out by Hochster in [7].

2. COLUMN GENERATORS

The simple Pascal pyramid has column generators, when it is double left-justified, which are

$$G_{m,n} = \frac{x^{m+n}\binom{m+n}{n}}{(1-x)^{m+n+1}} .$$

These columns can be shifted up and down with parameters p and q. The parameter p determines the alignment of the left-most slice of columns and the parameter q determines the alignment of the slices relative to that left-most slice. Now the modified simple column generators are

$$G^*_{m,n} = \frac{x^{pm+qn}\binom{m+n}{n}}{(1-x)^{m+n+1}} .$$

We desire to get the generating function of the planar section sum sequence. Each such planar section now has summands which are all multiplied by the same power of x. For instance,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^{m+n}\binom{m+n}{n}}{(1-x)^{m+n+1}} = \sum_{n=0}^{\infty} \frac{x^n}{(1-x)^{n+1}} \left\{ \sum_{m=0}^{\infty} \frac{x^m}{(1-x)^m} \binom{m+n}{n} \right\} .$$

But

$$\sum_{m=0}^{\infty} \binom{m}{n} z^m = \frac{z^n}{(1-z)^{n+1}} ,$$

so that

$$\sum_{m=0}^{\infty} \binom{m+n}{n} z^m = \frac{1}{(1-z)^{n+1}} .$$

Thus

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^{m+n}}{(1-x)^{m+n+1}} \binom{m+n}{n} = \sum_{n=0}^{\infty} \frac{x^n}{(1-x)^{n+1}} \cdot \frac{1}{\left(1 - \frac{x}{1-x}\right)^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{(1-2x)^{n+1}} = \frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$$

which was to be expected as each planar section contains the numbers in the expansion $(1 + 1 + 1)^n$.

We next let p and q be utilized.

$$G = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G^*_{m,n} = \frac{1}{1 - x - x^p - x^q} .$$

Here clearly when $p = 2$ and $q = 3$ we get

$$G = \frac{1}{1 - x - x^2 - x^3} = \sum_{n=0}^{\infty} T_{n+1} x^n ,$$

the generating function for the Tribonacci numbers,

$$T_0 = 0, \quad T_1 = 1, \quad T_2 = 1, \quad \text{and} \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n .$$

If, on the other hand, we set $p = 1$ and $q = 2$, then

$$G = \frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} P_{n+1} x^n ,$$

the generating function for the Pell numbers, $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. One can get even more out of this.

Let $p = t + 1$ and $q = 2t + 1$; then,

$$G = \frac{1}{1 - x - x^{t+1} - x^{2t+1}} = \sum_{n=0}^{\infty} u(n; t, 1)x^n$$

the generating function for the generalized Fibonacci numbers of Harris and Styles [2] applied to the trinomial triangle whose coefficients are induced by the expansions

$$(1 + x + x^2)^n, \qquad n = 0, 1, 2, \cdots .$$

See also Hoggatt and Bicknell [3].

Consider

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} \frac{x^{mp+qn}\binom{m+n}{n}}{(1 - x)^{m+n+1}}\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\left[\frac{x^p}{1 - x}\right]^m \binom{m+n}{n}\right)\frac{x^{qn}}{(1 - x)^{n+1}} .$$

Let us now take every r^{th} slice in the general p, q case

$$\sum_{n=0}^{\infty} \frac{x^{qnr}}{(1 - x - x^p)^{rn+1}} = \frac{1}{1 - x - x^p} \cdot \frac{1}{1 - \dfrac{x^{qr}}{(1 - x - x^p)^r}}$$

$$\frac{(1 - x - x^p)^{r-1}}{(1 - x - x^p)^r - x^{rq}} = \frac{(1 - x - x^p)^{r-1}}{(1 - x - x^p)^r - x^{r+q'}} = \sum_{n=0}^{\infty} U(n; q', r)x^n ,$$

where $q' = r(q - 1)$, which is the generating function for the generalized Fibonacci numbers of Harris and Styles $U(n; q', r)$ as applied to the CONVOLUTION triangle of the number sequence $u(n; p - 1, 1)$ which are themselves generalized Fibonacci numbers of Harris and Styles in the binomial triangle. (See "Convolution Triangles for Generalized Fibonacci Numbers" [4].)

3. THE GENERAL CASE

In [5] Pascal's pyramid in standard position has as its elements in a horizontal plane the expansions of $(a + b + c)^n$, $n = 0, 1, 2, 3, \cdots$ with each planar section laid out as an equilateral lattice. In our configuration it is a right isosceles lattice.

The general column generator is

$$G^*_{m,n} = \frac{x^{pm+qn} b^m c^n \binom{m+n}{n}}{(1 - ax)^{m+n+1}}$$

and it is not difficult to derive that

$$G = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G^*_{m,n} = \frac{1}{1 - ax - bx^p - cx^q} .$$

Thus by selecting the five parameters one can get many other known generating functions.

Example 1. $a = 2, \; b = 2, \; c = -1, \; p = 2, \; q = 3,$

$$G = \frac{1}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} F_{n+1}F_{n+2}x^n .$$

Example 2. $a = 1, \; b + c = 1, \; p = q = 2,$ then

$$G = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n .$$

One notes that the condition $b + c = 1$ allows an infinitude of choices of integers b and c.

Example 3. Let

$$a = 3(1 - x^2), \quad b = 6, \quad c = -1, \quad p = 2, \quad \text{and} \quad q = 4,$$

then

$$G = \frac{1}{1 - 3x - 6x^2 + 3x^3 + x^4} = \sum_{m=0}^{\infty} \left(\begin{matrix} m + 3 \\ 3 \end{matrix}\right) x^m ,$$

where $\left(\begin{matrix} m \\ n \end{matrix}\right)$ are the Fibonomial coefficients. See H-78 and [8], or it can be written as

$$G = \sum_{m=0}^{\infty} \left(\frac{F_{m+1}F_{m+2}F_{m+3}}{1 \cdot 1 \cdot 2} \right) x^m .$$

The possibilities seem endless.

4. FURTHER RESULTS

Consider

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{x^{pm} b^m \binom{m+n}{n}}{(1-ax)^m} \right\} \frac{c^n x^{nq}}{(1-ax)^{n+1}} .$$

Now let's take every r^{th} slice:

$$G = \sum_{n=0}^{\infty} \frac{(cx^q)^{rn}}{(1 - ax - bx^p)^{rn+1}} = \frac{(1 - ax - bx^p)^{r-1}}{(1 - ax - bx^p)^r - c^r x^{r+q'}} ,$$

where $q' = q(r-1)$. If $c = 1$, $a = 2$, $b = -1$, $p' = r + q'$, and $p = 2$, then

$$G = \frac{(1-x)^{2r-2}}{(1-x)^{2r} - x^{2r+p'}} .$$

Recall from [1] and [3] that

$$H = \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n; p, q) x^n$$

for the generalized Fibonacci numbers in Pascal's triangle so that G is the generating function for $H/(1-x)$ or

$$G = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} u(k; p', 2r) \right\} x^n .$$

Another example: If $a = 1 + x$, $b = 1$, $p = 3$, $c = 1$, then

$$G = \frac{(1 - x - x^2 - x^3)^{r-1}}{(1 - x - x^2 - x^3)^r - x^{r+q'}} ,$$

[Continued on page 293.]