# SOME NEW NARCISSISTIC NUMBERS

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A Narcissistic number is one which can be represented as some function of its digits. For example,

$$153 = 1^3 + 5^3 + 3^3, \qquad 145 = 1! + 4! + 5!, \quad \text{and} \quad 2427 = 2^1 + 4^2 + 2^3 + 7^4$$

are narcissistic numbers. One special class of these numbers, represented by the first example above, are called Digital Invariants. These are integers which are equal to the sum of the $n^{th}$ powers of the digits of the integers. Extensive studies of digital invariants have been in progress during the past two years. Robert L. Patton, Sr., Robert L. Patton, Jr., and the author have completed the search for all digital invariants for $n^{th}$ powers up to $n = 15$ and will publish the results in the near future.

This short note reports on various narcissistic numbers other than digital invariants. An abbreviated form for these numbers is used in Table 1.

$$abc \cdots \quad \text{means} \quad 10^p a + 10^{p-1} b + 10^{p-2} c + \cdots + 10p + q ,$$

where $a, b, c, \cdots, q$ are the digits of the integer and the number of digits is $p + 1$. That is,

$$349 = 10^2 \cdot 3 + 10 \cdot 4 + 9 .$$

The general form is shown in the Table along with the known solutions, their discoverers, and some notes. Trivial solutions, 0 and 1, are not included.

The search for solutions to the first form

$$(abc \cdots = a^n + b^{n+1} + c^{n+2} + \cdots)$$

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shown in Table 1 is far from complete. If $n = 1$, a complete search would entail checking all integers less than 23 digits in length (more precisely, integers less than about $1.108 \times 10^{21}$). There are comparable, though larger, searches if $n > 1$. A WANG 700 Programmable Calculator took about five hours to find the list shown in Table 1.

The search for the second solution to the form

$$abc \cdots = a^a + b^b + c^c + \cdots$$

took about one hour on an IBM 360/50 Computer. The factorial and subfactorial forms were searched to check for the possibility of missed solutions. In less than 15 minutes on the IBM 360/50 Computer the solutions shown were confirmed to be the only ones.

A secondary search was made in isolated cases for recurring forms. For example:

$$\begin{aligned} 169: \quad 1! + 6! + 9! &= 36301 \\ 3! + 6! + 3! + 0! + 1! &= 1454 \\ 1! + 4! + 5! + 4! &= 169 \end{aligned}$$

or, briefly, digital factorial $169 \rightarrow 36301 \rightarrow 1454 \rightarrow 169$ (3 cycles). Similarly, digital factorial $871 \rightarrow 45361 \rightarrow 871$ (2 cycles),
$872 \rightarrow 45362 \rightarrow 872$ (2 cycles).

No other recurring forms for digital factorials were found, but the cycle search was limited to five or less. There are undoubtedly many others with a greater number of cycles.

A few recurring forms for the digital exponent form

$$(abc \cdots = a^a + b^b + c^c + \cdots)$$

were found by sheer trial and error on a WANG 700 Calculator. The initial integer in the following examples is the smallest member of the cycle series.

Digital exponent $288 \rightarrow 33554436 \rightarrow \cdots \rightarrow 140023 \rightarrow 288$ (58 cycles).

Digital exponent 3439 $\longrightarrow$ 387420799 $\longrightarrow \cdots \longrightarrow$ 53423 $\longrightarrow$ 3439 (52 cycles).
Digital exponent 50119 $\longrightarrow$ 387423618 $\longrightarrow \cdots \longrightarrow$ 33601354 $\longrightarrow$ 50119 (25 cycles).

Searching for interesting integers is obviously endless! I hope some readers will warm up their pencils, calculators, or computers and search further into the Table and report any new additions — including forms not shown here. (Notes and discoverers are shown on the following page.)

Table 1
NARCISSISTIC NUMBERS

| Form | Solutions | Discoverer | Notes |
|---|---|---|---|
| $abc\cdots = a^n + b^{n+1} + c^{n+2} + \cdots$ | $43 = 4^2 + 3^3$ | 3 | |
| | $63 = 6^2 + 3^3$ | 3 | |
| | $89 = 8^1 + 9^2$ | 8 | |
| | $135 = 1^1 + 3^2 + 5^3$ | 2 | |
| | $175 = 1^1 + 7^2 + 5^3$ | 3 | |
| | $518 = 5^1 + 1^2 + 8^3$ | 3 | |
| | $598 = 5^1 + 9^2 + 8^3$ | 2 | |
| | $1306 = 1^1 + 3^2 + 0^3 + 6^4$ | 8 | |
| | $1676 = 1^1 + 6^2 + 7^3 + 6^4$ | 8 | B |
| | $2427 = 2^1 + 4^2 + 2^3 + 7^4$ | 8 | |
| | $6714 = 6^3 + 7^4 + 1^5 + 4^6$ | 5 | |
| | $47016 = 4^2 + 7^3 + 0^4 + 1^5 + 6^6$ | 5 | |
| | $63760 = 6^3 + 3^4 + 7^5 + 6^6 + 0^7$ | 5 | |
| | $63761 = 6^3 + 3^4 + 7^5 + 6^6 + 1^7$ | 5 | |
| | $542186 = 5^2 + 4^3 + 2^4 + 1^5 + 8^6 + 6^7$ | 5 | |
| $abc = a^n + b^{n-1} + c^{n-2} + \cdots$ | $24 = 2^3 + 4^2$ | 7 | |
| | $332 = 3^5 + 3^4 + 2^3$ | 7 | |
| | $1676 = 1^5 + 6^4 + 7^3 + 6^2$ | 7 | B |
| $abc = a^a + b^b + c^c + \cdots$ | $3435 = 3^3 + 4^4 + 3^3 + 5^5$ | 8 | |
| | $438579088 = 4^4 + 3^3 + 8^8 + 5^5 + 7^7 + 9^9 + 0^0 + 8^8 + 8^8$ | 6 | A |
| $abc = a! + b! + c! + \cdots$ | $2 = 2!$ | | |
| | $145 = 1! + 4! + 5!$ | 8 | |
| | $40585 = 4! + 0! + 5! + 8! + 5!$ | 4 | |
| $abc = !a + !b + !c + \cdots$ | $148349 = 1! + !4 + !8 + !3 + !4 + !9$ | 1 | C |

Notes for this table are found on the following page.

A. Since $0^0$ is indeterminant, two assumed values were tested: $0^0 = 0$ and $0^0 = 1$. There are no solutions using $0^0 = 1$, so the solution (438579088) shown assumes $0^0 = 0$.

B. 1676 is most interesting: appearing in two places in this table!

C. $!n$ is the subfactorial $n$ and is given by the formula:

$$!n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\left(\frac{1}{n!}\right)\right]$$

so that $!0 = 0$, $!1 = 0$, $!2 = 1$, $!3 = 2$, $!4 = 9$, $!5 = 44$, and so on. The number shown in the Table is the only non-trivial solution for this form.

## DISCOVERERS

1. Ron S. Dougherty, in a letter to the author dated April 28, 1965. published in Mathematics on Vacation by J. S. Madachy (Scribner's Sons, 1966), page 167.
2. Dale Kozniuk, included in "Curious Number Relationships," Recreational Mathematics Magazine, No. 10, August 1962, page 42.
3. J. A. H. Hunter, "Number Curiosities," Recreational Mathematics Magazine, No. 13, February 1963, page 28.
4. Leigh Janes, discovered in 1964 and published in Mathematics on Vacation (see [1] above) without proper credit, inadvertently.
5. Joseph S. Madachy, discovered 1970 on WANG 700 Programmable Calculator.
6. Joseph S. Madachy, discovered 1970 on IBM 360/50 Computer.
7. Joseph S. Madachy, discovered 1971 on Hewlett-Packard 9100B Programmable Calculator.
8. Unknown.