ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-195 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California

Consider the array indicated below:

1	1						
1	2						
2	4	1	1				
5	9	3	4				
13	22	7	11	1	1		
34	56	16	27	5	6		
89	145	38	65	16	22	1	1
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

(i) Show that the row sums are F_{2n}, $n \geq 2$.

(ii) Show that the rising diagonal sums are the convolution of

$$\left\{F_{2n-1}\right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{u(n;\ 2,2)\right\}_{n=0}^{\infty} ,$$

the generalized numbers of Harris and Styles.

H-196 Proposed by J. B. Roberts, Reed College, Portland, Oregon.

(a) Let A_0 be the set of integral parts of the positive integral multiples of τ, where

$$\tau = \frac{1 + \sqrt{5}}{2} ,$$

and let A_{m+1}, $m = 0, 1, 2, \cdots$, be the set of integral parts of the numbers $n\tau^2$ for $n \in A_m$. Prove that the collection of Z^+ of all positive integers is the disjoint union of the A_j.

(b) Generalize the proposition in (a).

H-197 Proposed by Lawrence Somer, University of Illinois, Urbana, Illinois.

Let $\{u_n^{(t)}\}_{n=1}^{\infty}$ be the t-Fibonacci sequences with positive entries satisfying the recursion relationship:

$$u_n^{(t)} = \sum_{i=1}^{t} u_{n-i} .$$

Find

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \frac{u_{n+1}^{(t)}}{u_n^{(t)}} .$$

SOLUTIONS

HYPER-TENSION

H-185 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$(1 - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \binom{2k}{k} (1 - v)^{n-k} {}_2F_1[-k; n + k + 1; k + 1; x] ,$$

where ${}_2F_1[a, b; c; x]$ denotes the hypergeometric function.

Solution by the Proposer.

We start with the identity

$$\sum_{r,s=0}^{\infty} \frac{(2r + 3s)!}{r!s!(r + 2s)!} \frac{(y - z)^r z^s}{(1 + y)^{2r+3s+1}} = \frac{1}{1 - y - z} .$$

Now put $y = u + v$, $z = v$, so that

$$(*) \qquad \sum_{r,s=0}^{\infty} \frac{(2r + 3s)!}{r!s!(r + 2s)!} \frac{u^r v^s}{(1 + u + v)^{2r+3s+1}} = \frac{1}{1 - u - 2v} .$$

The right-hand side of (*) is equal to

$$\sum_{n=0}^{\infty} (u + 2v)^n ,$$

while the left-hand side

$$= \sum_{r,s=0}^{\infty} \frac{(2r + 3s)!}{r!s!(r + 2s)!} u^r v^s \sum_{k=0}^{\infty} (-1)^k \binom{2r + 3s + k}{k} (u + v)^k .$$

It follows that

$$(u + 2v)^n = \sum_{k=0}^{n} (u + v)^k \sum_{r+s=n-k} (-1)^k \binom{2r + 3s + k}{k} \frac{(2r + 3s)!}{r!s!(r + 2s)!} u^r v^s$$

$$= \sum_{r+s \le n} (-1)^{n-r-s} \frac{(r + 2s + n)!}{r!s!(r + 2s)!(n - r - s)!} u^r v^s (u + v)^{n-r-s}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \frac{(u + v)^{n-k}}{(n - k)!} \sum_{s=0}^{k} \frac{(s + n + k)!}{s!(k - s)!(k + s)!} u^{k-s} v^s .$$

Taking $u = 1$, $v = -x$, we get

$$(1 - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \frac{(n + k)!}{k!k!(n - k)!} (1 - v)^{n-k} {}_2F_1[-k, n + k + 1; k + 1; x] .$$

A CONGRUENCE IN ITS PRIME

H-186 Proposed by James Desmond, Florida State University, Tallahassee, Florida.

The generalized Fibonacci sequence is defined by the recurrence relation

$$U_{n-1} + U_n = U_{n+1} ,$$

where n is an integer and U_0 and U_1 are arbitrary fixed integers.

For a prime p and integers n, r, s and t, show that

$$U_{np+r} \equiv U_{sp+t} \pmod{p} ,$$

if $p \equiv \pm 1 \pmod 5$ and $n + r = s + t$, and that

$$U_{np+r} \equiv (-1)^{r+t} U_{sp+t} \pmod p$$

if $p \equiv \pm 2 \pmod 5$ and $n - r = s - t$.

Solution by the Proposer.

We have from Hoggatt and Ruggles, "A Primer for the Fibonacci Sequence — Part III," Fibonacci Quarterly, Vol. 1, No. 3, 1963, p. 65, and by Fermat's theorem, that

$$F_{np+r} = \sum_{i=0}^{p} \binom{p}{i} F_{i+r} F_n^i F_{n-1}^{p-i} \equiv F_r F_{n-1}^p + F_{p+r} F_n^p \equiv F_r F_{n-1} + F_{p+r} F_n \pmod p$$

for all n and r. From I. D. Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, Vol. 1, No. 2, 1963, p. 79, we have that

$$F_{i+j} = F_{i+1} F_j + F_i F_{j-1}$$

for all i and j. Therefore,

$$F_{np+r} \equiv F_r F_{n-1} + F_{r+1} F_p F_n + F_r F_{p-1} F_n \pmod p$$

for all n and r. We have from Hardy and Wright, Theory of Numbers, Oxford University Press, London, 1954, p. 150, that

$$F_{p-1} \equiv 0 \pmod p \quad \text{and} \quad F_p \equiv 1 \pmod p$$

if $p \equiv \pm 1 \pmod 5$, and that

$$F_{p+1} \equiv 0 \pmod p \quad \text{and} \quad F_p \equiv 1 \pmod p$$

if $p \equiv \pm 2 \pmod 5$. Let $p \equiv \pm 1 \pmod 5$ and $n + r = s + t$. Then

$$F_{np+r} \equiv F_r F_{n-1} + F_{r+1} F_n \equiv F_{r+n} \pmod p$$

for all n and r. Therefore

$$F_{sp+t} \equiv F_{s+t} \equiv F_{n+r} \equiv F_{np+r} \pmod p.$$

It is easily verified by induction that

$$U_n = U_1F_n + U_0F_{n-1}$$

for all n. Therefore

$$U_{np+r} \equiv U_1 F_{np+r} + U_0 F_{np+r-1} \equiv U_1F_{sp+t} + U_0F_{sp+t-1} \equiv U_{sp+t} \pmod{p} .$$

Now, let $p \equiv +2 \pmod{5}$ and $n - r = s - t$. From page 77 of the reference to Ruggles, we have

$$F_{i+j} - F_iL_j = (-1)^{j+1}F_{i-j}$$

for all i and j. Therefore

$$\begin{aligned} F_{np+r} &\equiv F_r F_{n-1} - F_{r+1} F_n + F_r F_n \equiv F_r F_{n-1} + F_{r+1} F_n - 2F_{r+1} F_n + F_rF_n \\ &\equiv F_{r+n} - L_r F_n \equiv (-1)^{r+1} F_{n-r} \pmod{p} \end{aligned}$$

for all n and r. Thus

$$(-1)^{r+t} F_{sp+t} \equiv (-1)^{r+t}(-1)^{t+1} F_{s-t} \equiv (-1)^{r+1} F_{n-r} \equiv F_{np+r} \pmod{p} .$$

Hence

$$\begin{aligned} U_{np+r} &\equiv U_1 F_{np+r} + U_0 F_{np+r-1} \equiv U_1(-1)^{r+t}F_{sp+t} + U_0(-1)^{r-1+t-1} F_{sp+t-1} \\ &\equiv (-1)^{r+t} (U_1F_{sp+t} + U_0 F_{sp+t-1}) \equiv (-1)^{r+t}U_{sp+t} \pmod{p}. \end{aligned}$$

FIBONACCI IS A SQUARE

H-187 Proposed by Ira Gessel, Harvard University, Cambridge, Massachusetts.

Problem: Show that a positive integer n is a Fibonacci number if and only if either $5n^2 + 4$ or $5n^2 - 4$ is a square.

Solution by the Proposer.

Let $F_0 = 0$, $F_1 = 1$, $F_{r+1} = F_r + F_{r-1}$ be the Fibonacci series and $L_0 = 2$, $L_1 = 1$, $L_{r+1} = L_r + L_{r-1}$ be the Lucas series. It is well known that

$$(-1)^r + F_r^2 = F_{r+1} F_{r-1} \tag{1}$$

$$L_r = F_{r+1} + F_{r-1} \tag{2}$$

Subtracting four times the first from the square of the second equation, we have

$$L_r^2 - 4(-1)^r - 4F_r^2 = (F_{r+1} - F_{r-1})^2 = F_r^2 ,$$

whence

$$5F_r^2 + 4(-1)^r = L_r^2 .$$

Thus if n is a Fibonacci number, either $5n^2 + 4$ or $5n^2 - 4$ is a square.

I have two proofs of the converse.

First Proof. We use the theorem (Hardy and Wright, An Introduction to the Theory of Numbers, p. 153) that if p and q are integers, x is a real number, and $|(p/q) - x| < 1/2q^2$, then p/q is a convergent to the continued fraction for x, and that (Hardy and Wright, p. 148) the convergents to the continued fraction for $(1 + \sqrt{5})/2$ in lowest terms are F_{r+1}/F_r.

Assume that $5n^2 \pm 4 - m^2$. Then since m and n have the same parity, $k = (m + n)/2$ is an integer. Then substituting $m = 2k - n$ in $5n^2 \pm 4 = m^2$, we get $k^2 - kn - n^2 = \pm 1$, so that k and n are relatively prime and

$$\pm 1/n^2 = (k/n)^2 - (k/n) - 1 = [(k/n) - (\sqrt{5} + 1)/2][(k/n) + (\sqrt{5} - 1)/2] .$$

Thus

$$|(k/n) - (\sqrt{5} + 1)/2| = 1/n^2 \, |(k/n) + (\sqrt{5} - 1)/2| .$$

Since 1 is a Fibonacci number, we may assume $n \geq 2$. Then

$$(2k - n)^2 = m^2 \geq 5n^2 - 4 = 4n^2 + (n^2 - 4) \geq 4n^2 ,$$

so $2k - n \geq 2n$, whence $k/n \geq 3/2$. Thus $(k/n) + (\sqrt{5} - 1)/2 > 2$, so by the two theorems quoted above, $k/n = F_{r+1}/F_r$ for some r, and since both fractions are reduced, $n = F_r$.

Second Proof. Assume $5n^2 \pm 4 = m^2$. Then $m^2 - 5n^2 = \pm 4$, so

$$\frac{m + \sqrt{5}n}{2} \cdot \frac{m - \sqrt{5}n}{2} = \pm 1 ,$$

and since m and n have the same parity,

$$\frac{m + \sqrt{5}n}{2} \quad \text{and} \quad \frac{m - \sqrt{5}n}{2}$$

are integers in $Q(\sqrt{5})$, where Q is the rationals, and since their product is ± 1, they are units. It is well known (Hardy and Wright, p. 221) that the only integral units of $Q(\sqrt{5})$ are of the form $\pm x^{\pm r}$, where $x = (1 + \sqrt{5})/2$.

Then we have

$$(m + \sqrt{5}n)/2 = x^r = \frac{1}{2}\left[(x^r + y^r) + \frac{x^r - y^r}{\sqrt{5}} \cdot \sqrt{5}\right],$$

where $y = -1/x$. Now $x^r + y^r = L_r$ and

$$(x^r - y^r)/\sqrt{5} = F_r$$

(Hardy and Wright, p. 148). Thus

$$\frac{1}{2}(m + \sqrt{5}n) = \frac{1}{2}(L_r + \sqrt{5}F_r),$$

so $n = F_r$.

SUM SERIES

H-189 Proposed by L. Carlitz, Duke University, Durham, North Carolina (Corrected).

Show that

$$\sum_{r,s=0}^{\infty} \frac{(2r+3s)!}{r!s!(r+2s)!} \frac{(a-by)^r b^s y^{r+2s}}{(1+ay)^{2r+3s+1}} = \frac{1}{1-ay-by^2}$$

Solution by the Proposer.

Put

$$\frac{1}{1-ax-bx^2} = \sum_{m=0}^{\infty} G_m x^m,$$

so that

$$\frac{1}{(1-ax-bx^2)(1-y)} = \sum_{m,n=0}^{\infty} G_m x^m y^n .$$

Replacing y by $x^{-1}y$ this becomes

$$\frac{1}{(1-ax-bx^2)(1-x^{-1}y)} = \sum_{m,n=0}^{\infty} G_m x^{m-n} y^n .$$

Hence that part of the expansion of

$$\frac{1}{(1-ax-bx^2)(1-x^{-1}y)}$$

that is independent of x is equal to

$$(*) \qquad \frac{1}{1 - ay - by^2}$$

On the other hand, since

$$(1 - ax - bx^2)(1 - x^{-1}y) = (1 + ay) - x(a - by) - bx^2 - x^{-1}y,$$

we have

$$(1 - ax - bx^2)^{-1}(1 - x^{-1}y)^{-1} = \sum_{k=0}^{\infty} \frac{[x(a - by) + bx^2 + x^{-1}y]^k}{(1 + ay)^{k+1}}$$

$$= \sum_{r,s,t=0}^{\infty} \frac{(r + s + t)!}{r!s!t!} \frac{(a - by)^r b^s y^t}{(1 + ay)^{r+s+t+1}} x^{r+2s-t} .$$

The part of this sum that is independent of x is obtained by taking $t = r + 2s$. We get

$$\sum_{r,s=0}^{\infty} \frac{(2r + 3s)!}{r!s!(r + 2s)!} \frac{(a - by)^r b^s y^{r+2s}}{(1 + ay)^{2r+3s+1}} .$$

Since this is equal to (*), we have proved the stated identity.

IT'S A MOD WORLD

H-190 Proposed by H. H. Ferns, Victoria, British Columbia.

Prove the following

$$2^r F_n \equiv n \pmod 5$$
$$2^r L_n \equiv 1 \pmod 5 ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, and r is the least residue of $n - 1 \pmod 4$.

Solution by the Proposer.

In an unpublished paper by the proposer, it is shown that

$$2^{n-1} F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} 5^k .$$

Hence

$$2^{n-1}F_n = n + \sum_{k=1}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} 5^k .$$

Thus

$$2^{n-1}F_n \equiv n \pmod{5} .$$

Let $n - 1 = 4m + r$, where $0 \le r < 4$. Then

$$2^{n-1}F_n = 2^{4m+r}F_n = 2^{4m}2^rF_n \equiv n \pmod{5} .$$

But

$$2^{4m} = (2^4)^m \equiv 1 \pmod{5} .$$

Hence

$$2^rF_n \equiv n \pmod{5} .$$

To prove

$$2^rL_n \equiv 1 \pmod{5} ,$$

use

$$2^{n-1}L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} 5^k$$

(which is derived in the same paper) and proceed as above.

JUST SO MANY TWO'S

H-192 Proposed by Ronald Alter, University of Kentucky, Lexington, Kentucky.

If

$$c_n = \sum_{j=0}^{3n+1} \binom{6n+3}{2j+1} (-11)^j ,$$

prove that

$$c_n = 2^{6n+3} \cdot N, \qquad (N \text{ odd}, \ n \ge 0) .$$

Solution by the Proposer.

In the sequence

$$b_k = b_{k-1} - 3b_{k-2}, \qquad (k \ge 1, \ b_1 = b_2 = 1) ,$$

it is easy to show that 2 is the highest power of 2 that divides b_k if and only if $k \equiv 3 \pmod{6}$. Also, by deriving the appropriate Binet formula, it follows that

$$b_k = \frac{1}{\sqrt{-11}}\left\{\left(\frac{1+\sqrt{-11}}{2}\right)^k - \left(\frac{1-\sqrt{-11}}{2}\right)^k\right\}, \qquad k \ge 1.$$

Thus

$$b_k = \frac{1}{2^{k-1}} \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2j+1}(-11)^j, \qquad k \ge 1.$$

The desired result follows by observing

$$b_{6n+3} = \frac{1}{2^{6n+2}}\, c_n \quad .$$

Editorial Note: Please submit solutions for any of the problem proposals. We need fresh blood!

A GOLDEN SECTION SEARCH PROBLEM

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After tiring of using numerous quadratic functions as objective functions for examples in my mathematical programming course, I posed the following problem for myself: Design a unimodal function over the (0,1) interval which is concave, has a maximum in the interior of (0,1), and is not a quadratic function. The purpose was to demonstrate numerically the golden section search.*

My first thoughts were to add two functions which are concave over the (0,1) interval with the property that one goes to $-\infty$ at 0 and the other goes to $-\infty$ at 1. My two initial choices were $\log x$ and $1/(x-1)$. The golden section search starts at the two points $x_1 = 1 - (1/\phi)$ and $x_2 = 1/\phi$ where $\phi = (1+\sqrt{5})/2$. After searching with 8 points, I noticed that the interval of uncertainty still contained the first search point so I thought it about time to find the location of the maximum analytically. I was dumfounded to discover that if I continued indefinitely with the search my interval of uncertainty would still contain the initial search point.

*Douglas J. Wilde, Optimum Seeking Methods, Prentice Hall, Inc. (1964).