YE OLDE FIBONACCI CURIOSITY SHOPPE

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In the good old days when Jekuthiel Ginsburg was Editor of Scripta Mathematica, there were many brief items of interesting mathematics, some with proof, some without, contributed by a wide variety of people. Some of these were labeled curiosities; others without being tagged as such were evidently in the same category. A fair amount of this material dealt with Fibonacci sequences. We offer a few for-instances translated into symbolism more familiar to readers of the Fibonacci Quarterly.

Charles W. Raine [1] noted that if four consecutive Fibonacci numbers are taken, then the product of the extreme terms can be used as one leg of a Pythagorean triangle, twice the product of the mean terms as the other, to give a hypotenuse which is a Fibonacci number whose index is the sum of the indices of the terms in one of the sides. For example, if

$$F_6 = 8, \quad F_7 = 13, \quad F_8 = 21, \quad F_9 = 34$$
$$a = 8 \times 34 = 272; \qquad b = 2 \times 13 \times 21 = 546;$$
$$c = \sqrt{272^2 + 546^2} = 610 = F_{15} .$$

Harlan L. Umansky [2] following up on Raine's idea extended the result to a generalized Fibonacci sequence. The sides of the Pythagorean triangle in this case would be given by:

$$a = T_k T_{k+3}; \qquad b = 2T_{k+1}T_{k+2}; \qquad c = b + T_k^2 \quad \text{or} \quad c = T_{k+1}^2 + T_{k+2}^2 .$$

For example, using the sequence 1, 4, 5, 9, 14, 23, $\cdots$ and taking the four values 5, 9, 14, 23, $a = 5 \times 23 = 115$; $b = 2 \times 9 \times 13 = 252$.

$$c = \sqrt{115^2 + 252^2} = 277 ,$$

while $$b + T_k^2 = 252 + 5^2 = 277 \quad \text{and} \quad 9^2 + 14^2 = 277.$$

Gershon Blank [3] pointed out that

$$(F_n + F_{n+6})F_k + (F_{n+2} + F_{n+4})F_{k+1} = L_{n+3}L_{k+1} .$$

For example, if $n = 5$, $k = 4$, $(5 + 89) \times 3 + (13 + 34) \times 5 = 517$, while $47 \times 11 = 517$.

A note signed G. (evidently J. Ginsburg himself) [4] quoted the cubic relation given by Dickson

$$F_{n+1}^2 + F_n^3 - F_{n-1}^3 = F_{3n}$$

and offered a second

$$F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = 3F_{3n} .$$

Thus for $n = 5$,

$$13^3 - 3 \times 5^3 + 2^3 = 1830 = 3F_{15} .$$

Fenton Stancliff [5] (A Curious Property of a_{11}) showed the following arrangement for finding the value of $1/89 = 0.011235955056\cdots$

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1/89 = 0.0112358
              13
               21
                34
                 55
                  89
                  144
                   233
                    377
       ----------------
       0.011235955056107
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P. Schub [6] (A Minor Fibonacci Curiosity) offered the relation:

$$5F_n^2 + 4(-1)^n = L_n^2 .$$

G. Candido [7] produced a fourth-power relation:

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) = (F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2$$

When $n = 5$, this becomes

$$2(5^4 + 8^4 + 13^4) = 66564$$

while

$$(5^2 + 8^2 + 13^2)^2 = 66564 .$$

Royal V. Heath [8] noted that the sum of ten consecutive Fibonacci numbers is divisible by 11 with a cofactor the seventh of the ten quantities. The sum of fourteen consecutive Fibonacci numbers is divisible by 29 with the cofactor the ninth; etc.

Harlan L. Umansky [9] (Curiosa, Zero Determinants) offered the following. For any series a, d, a + d, a + 2d, 2a + 3d, etc., if N^2 consecutive terms ($N \geq 2$) are taken and placed consecutively in the columns of a determinant, the value of the determinant is zero. Thus:

$$\begin{vmatrix} 1 & 11 & 76 & 521 \\ 3 & 18 & 123 & 843 \\ 4 & 29 & 199 & 1364 \\ 7 & 47 & 322 & 2207 \end{vmatrix} = 0$$

These examples are sufficient to give the general flavor of the contributions of those days. I believe there must be people today likewise who would be happy to express themselves in this fashion once more. Recently, in a private communication, William H. Huff stated that he had discovered the following. Add up any number of consecutive Fibonacci numbers; then add the second to the result; the final answer is always a Fibonacci number. The same seems to hold for any Fibonacci sequence starting with two values a, b.

Another curiosity is the fact that the sum of the squares of consecutive Fibonacci numbers is always divisible by $F_{10} = 15$. What about other sums of squares either of the Fibonacci sequence proper or other Fibonacci sequences? Are there similar results for sums of cubes, fourth powers, etc.?

This article is an invitation to our readers to engage in the type of activity that used to be featured by Scripta Mathematica. While it is probably true that one man's "curiosity" is another man's formula, article, or thesis, we shall content ourselves by defining a "curiosity" as follows: A fact or relation that seems interesting and calculated to arouse intellectual curiosity which is offered without proof for the consideration of the readers of the Fibonacci Quarterly. If you want something to appear in this department, be sure to label it FIBONACCI CURIOSITY.

REFERENCES

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9. H. L. Umansky, "Curiosa. Zero Determinants," Scripta Mathematica, xxii, No. 1, March 1956, p. 88.