PERFECT N-SEQUENCES FOR N, N+1, AND N+2

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Frank S. Gillespie and W. R. Utz [1] define a (generalized) perfect n-sequence for m (where $n \geq 2$, $m \geq 2$) to be a sequence of length mn in which each of the integers 1, 2, 3, $\cdots$, m occurs exactly n times and between any two occurrences of the integer x there are x entries. Examples of perfect 2-sequences are numerous: 3 1 2 1 3 2 for $m = 3$ and 4 1 3 1 2 4 3 2 for $m = 4$ are the simplest. However, the author knows of no perfect n-sequence if $n > 2$.

No perfect n-sequence for m exists if $m \leq n$ [1]. (This is a direct corollary of Lemma 1, below.) It will be proved here that no perfect n-sequence for m exists if $m = n$, $m = n + 1$, or $m = n + 2$ (except for the perfect 2-sequences for 3 and 4), extending the result slightly.

In a perfect n-sequence for m, if x is an integer and $1 \leq x \leq m$, then there are n x's in the sequence. The positions in the sequence will be numbered, in order, starting at the left, 1, 2, 3, $\cdots$, mn. Let "p(x,i)" mean "the position of the i^{th} occurrence of the integer x". The first occurrence of an integer will have special significance; let $P_x = p(x,1)$.

Example. In the sequence 1 7 1 2 6 4 2 5 3 7 4 6 3 5, $p(6,1) = P_6 = 5$, $p(4,2) = 11$, $P_2 = 4$, etc.

Note that p(x,i) is meaningful if $1 \leq x \leq m$ and $1 \leq i \leq n$, and P_x is meaningful if $1 \leq x \leq m$.

In a perfect n-sequence for m

$$(1) \qquad p(x,i) = P_x + (x+1)(i-1) \qquad (1 \leq x \leq m;\ 1 \leq i \leq n)$$

which follows from the recursive formula (for $i \geq 2$)

$$(2) \qquad p(x,i) = p(x, i-1) + (x+1)\,.$$

Theorem 1. There is no perfect n-sequence for n.

Proof. Assume such a sequence exists. Then it has n^2 entries. Also

$$p(n,n) = P_n + (n+1)(n-1) = P_n + n^2 - 1$$

so that P_n must be 1.

It is impossible that $1 \leq P_{n-1} \leq n$; otherwise $p(n-1, P_{n-1})$ and $p(n, P_{n-1})$ are meaningful and

$$p(n - 1, P_{n-1}) = P_{n-1} + nP_{n-1} - n = p(n, P_{n-1})$$

using (1) and $P_n = 1$. But this is impossible since an n and an $n - 1$ cannot occupy the same position.

It is impossible that $n + 1 \le P_{n-1}$; otherwise $p(n - 1, n) \ge n^2 + 1$, but the largest position is n^2.

Now $1 \le n - 1 \le n$ (since $n \ge 2$) so that P_{n-1} is a positive integer, and we have a contradiction.

Theorem 2. There is no perfect n-sequence for $n + 1$, except the perfect 2-sequence for 3.

Proof. Assume such a sequence exists. Then there are $n(n + 1) = n^2 + n$ entries. Also,

$$p(n + 1, n) = P_{n+1} + n^2 + n - 2$$

so that either $P_{n+1} = 1$ or $P_{n+1} = 2$. If $P_{n+1} = 2$, then $p(n + 1, n) = n^2 + n$, the last position, but since a perfect sequence taken in reverse order is still a perfect sequence, this case is symmetrical to the case $P_{n+1} = 1$. Hence only the case $P_{n+1} = 1$ need be considered.

It is impossible that $1 \le P_n \le n$; otherwise $p(n, P_n) = p(n + 1, P_n)$. It is impossible that $n + 2 \le P_n$; otherwise $p(n, n) \ge n^2 + n + 1$. Therefore the only possibility is $P_n = n + 1$. Now we have $P_{n+1} = 1$ and $P_n = n + 1$.

It is impossible that $1 \le P_{n-1} \le n - 1$; otherwise $p(n - 1, P_{n-1} + 1) = p(n, P_{n-1})$. It is impossible that $n + 1 \le P_{n-1} \le 2n$; otherwise $p(n - 1, P_{n-1} - n) = p(n, P_{n-1} - n)$. It is impossible that $2n + 1 \le P_{n-1}$; otherwise $p(n - 1, n) \ge n^2 + n + 1$. Therefore the only possibility is $P_{n-1} = n$.

It is impossible that $1 \le P_{n-2} \le n - 1$; otherwise

$$p(n - 2, P_{n-2} + 1) = p(n - 1, P_{n-2}).$$

It is impossible that $n \le P_{n-2} \le 2n - 1$; otherwise

$$p(n - 2, P_{n-2} - n + 1) = p(n - 1, P_{n-2} - n + 1) .$$

It is impossible that $2n \le P_{n-2} \le 3n - 2$; otherwise

$$p(n - 2, P_{n-2} - 2n + 1) = p(n - 1, P_{n-2} - 2n + 2) .$$

It is impossible that $P_{n-2} = 3n - 1$; otherwise $p(n - 2, n) = p(n, n)$. It is impossible that $3n \le P_{n-2}$; otherwise $p(n - 2, n) \ge n^2 + n + 1$. If $n \ne 2$, then $1 \le n - 2 \le n$ and P_{n-2} is a positive integer, a contradiction. The only possibility is therefore $n = 2$.

From these two theorems some patterns can be seen. They are formulated in the following lemmas.

Lemma 1. In a perfect n-sequence for m, if $1 \le n - r \le m$, then

$$P_{n-r} \le mn - n^2 + nr - r + 1.$$

In particular, in a perfect n-sequence for $n + i$, $P_{n-r} \le nr + in - r + 1$.

Proof. If $P_{n-r} > mn - n^2 + nr - r + 1$, then $p(n - r, n) > mn$, which is impossible since the largest position is mn.

Lemma 2. In a perfect n-sequence for m, if P_x and P_{x+1} are meaningful, then it is impossible that

$$P_{x+1} + (i - 1)x + (2i - 2) \le P_x \le P_{x+1} + (i - 1)x + (i - 2) + n \tag{3}$$

for any integer $i \ge 1$, or that

$$P_{x+1} + (i - 1)x + (i - 1) \le P_x \le P_{x+1} + (i - 1)x + (2i - 3) + n \tag{4}$$

for any integer $i \le 1$.

Proof. Assuming (3) to hold (with $i \ge 1$), we have

$$P_{x+1} + (i - 1)x + (2i - 2) \le P_x \tag{5}$$

$$P_x \le P_{x+1} + (i - 1)x + (i - 2) + n. \tag{6}$$

It follows from (5) and (6), respectively, that

$$P_{x+1} + (i - 1)x + (i - 1) \le P_x \tag{7}$$

$$P_x \le P_{x+1} + (i - 1)x + (2i - 3) + n. \tag{8}$$

From (5) and (8) follows

$$1 \le P_x - P_{x+1} - ix + x - 2i + 3 \le n, \tag{9}$$

and from (7) and (6) follows

$$1 \le P_x - P_{x+1} - ix + x - i + 2 \le n. \tag{10}$$

Finally, we have

$$p(x, P_x - P_{x+1} - ix + x - 2i + 3) = p(x + 1, P_x - P_{x+1} - ix + x - i + 2), \tag{11}$$

which is meaningful by (9) and (10). But (11) is obviously false, hence (3) cannot hold if $i \geq 1$. The proof of the second half is identical.

Corollary to Lemma 2. If P_x and P_{x+1} are meaningful, then either

$$P_{x+1} + (i - 1)x + (i - 2) + n < P_x < P_{x+1} + ix + 2i$$

for some $i \geq 1$, or

$$P_{x+1} + (i - 1)x + (2i - 3) + n < P_x < P_{x+1} + ix + i$$

for some $i \leq 0$.

Theorem 3. There is no perfect n-sequence for $n + 2$, except the perfect 2-sequence for 4.

Proof. This sequence has $n^2 + 2n$ entries. By Lemma 1, the only possibilities for P_{n+2} are (case I) $P_{n+2} = 1$, (case II) $P_{n+2} = 2$, and (case III) $P_{n+2} = 3$.

Case I. $P_{n+2} = 1$. By Lemma 1 and the Corollary to Lemma 2, the only possibilities for P_{n+1} are (case IA) $P_{n+1} = n + 1$ and (case IB) $P_{n+1} = n + 2$.

Case IA. $P_{n+1} = n + 1$. By the lemmas, the only possibilities for P_n are 1, $n - 1$, n, $n + 1$, and $2n + 1$. But $P_n = 1$ is impossible; otherwise $p(n,1) = p(n + 2,1)$; $P_n = n + 1$ is impossible; otherwise $p(n,1) = p(n + 1,1)$. Therefore there are three possibilities: (case IA1) $P_n = n - 1$, (case IA2) $P_n = n$, and (case IA3) $P_n = 2n - 1$.

Case IA1. $P_n = n - 1$. The possibilities for P_{n-1} are $n - 2$, $2n - 1$, $3n - 1$, and $3n$. But n even is impossible; otherwise $p(n, n/2) = p(n + 2, n/2)$; so n is odd; $P_{n-1} = n - 2$ is impossible; otherwise $p(n - 1, (n + 1)/2) = p(n + 1, (n - 1)/2)$; $P_{n-1} = 3n - 1$ is impossible; otherwise $p(n - 1, n) = p(n + 1, n)$; $P_{n-1} = 3n$ is impossible; otherwise

$$p(n - 1, (n - 1)/2) = p(n + 1, (n + 1)/2) ;$$

Therefore $P_{n-1} = 2n - 1$. The possibilities for P_{n-2} are $n - 1$, $4n - 2$, and $4n - 1$. But $P_{n-2} = n - 1$ is impossible; otherwise $p(n - 2,1) = p(n,1)$; $P_{n-2} = 4n - 2$ is impossible; otherwise (noting that $1 \leq (n + 3)/2 \leq n$ since $n \geq 2$ and n is odd)

$$p(n - 2, (n - 1)/2) = p(n, (n + 3)/2) ;$$

$P_{n-2} = 4n - 1$ is impossible; otherwise $p(n - 2,1) = p(n - 1,3)$. But $1 \leq n - 2 \leq n$ (since $n \geq 2$ and n odd) so that P_{n-2} is a positive integer, which is a contradiction. Therefore case IA1 is impossible.

This first case indicates the methods used. The others are treated similarly. The other cases are:

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