THE COEFFICIENTS OF cosh x/cos x

J. M. GANDHI and V. S. TANEJA
Department of Mathematics, Western Illinois University, Macomb, Illinois

1. Gandhi [3] defined a set of rational integral coefficients S_{2n} by the generating function

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} \frac{S_{2n}x^{2n}}{(2n)!} . \tag{1}$$

The coefficients S_{2n} were the subject of much investigation by Carlitz [1], [2], Gandhi [4], [5], Gandhi and Ajaib Singh [6], Krick [7], Raab [8] and Salie [9]. In the present note we prove that

$$S_{4n+2} \equiv 52 \pmod{100} \quad \text{for} \quad n > 0 . \tag{2}$$

The proof of (2) involves some elementary but interesting results.

2. Gandhi and Ajaib Singh [6] proved that

$$S_{4n+2} = \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} S_{4n+2-4r} + 2^{4n+1} . \tag{3}$$

Assume that (2) is true for any $n > 0$ and we shall prove that it is true for $n + 5$. Since S_6, S_{10}, $\cdots$, $S_{4n-2} \equiv 52 \pmod{100}$, and $S_2 = 2$, Eq. (3) yields

$$S_{4n+2} \equiv 52 \sum_{r=1}^{n-1} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} + \binom{4n+2}{4n} (-1)^{n+1} 2^{2n+1} + 2^{4n+1} \pmod{100}$$

$$\equiv 52 \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} + \binom{4n+2}{4n} (-1)^{n+1} 2^{2n} [2 - 52] + 2^{4n+1} \pmod{100}.$$

Since $n > 0$, the second term on the right is divisible by 100 and therefore

$$S_{4n+2} \equiv 52 \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} + 2^{4n+1}$$

(4)
$$\equiv 104 \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r-1} + 2^{4n+1}$$

$$\equiv 2 \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} + 2^{4n+1} \pmod{100}$$

$$\equiv 2A + 2^{4n+1} \pmod{100} \quad ,$$

where

$$A = \sum_{r=1}^{n} \binom{4n+2}{4r} (-1)^{r+1} 2^{2r} \quad .$$

We now evaluate the sum for A. Let $\omega = (1 + i)/\sqrt{2}$, then it can be verified that $\omega^4 = -1$ and $\omega^8 = +1$, where $i = \sqrt{-1}$. Now

$$(1 + \omega x)^{4n+2} = \sum_{r=0}^{4n+2} \binom{4n+2}{r} \omega^r x^r \quad ,$$

and

$$(1 - \omega x)^{4n+2} = \sum_{r=0}^{4n+2} \binom{4n+2}{r} (-1)^r \omega^r x^r \quad .$$

Adding these two expansions we get

(5)
$$\frac{(1 + \omega x)^{4n+2} + (1 - \omega x)^{4n+2}}{2} = \sum_{r=0}^{2n+1} \binom{4n+2}{2r} \omega^{2r} x^{2r} \quad .$$

In (5) replace x by $\sqrt{-1}x$ to get

(6)
$$\frac{(1 + \sqrt{-1}\omega x)^{4n+2} + (1 - \sqrt{-1}\omega x)^{4n+2}}{2} = \sum_{r=0}^{2n+1} \binom{4n+2}{2r} (-1)^r \omega^{2r} x^{2r} \, .$$

Adding (5) and (6) and letting $x = \sqrt{2}$ it is easy to see that

(7) $$A = 1 - \tfrac{1}{4}\left[(1+\omega\sqrt{2})^{4n+2} + (1-\omega\sqrt{2})^{4n+2} + (1+\sqrt{-1}\,\omega\sqrt{2})^{4n+2} + (1-\sqrt{-1}\,\omega\sqrt{2})^{4n+2}\right].$$

Since $\omega\sqrt{2} = 1 + i$, Eq. (7) becomes

$$A = 1 - \tfrac{1}{4}\left[(2+i)^{4n+2} + (-1)^{4n+2} + (i)^{4n+2} + (2-i)^{4n+2}\right] = 1 - \tfrac{1}{4}\left[(3+4i)^{2n+1} + (3-4i)^{2n+1} - 2\right].$$

Using this expression for A, Eq. (4) becomes

(8) $$S_{4n+2} \equiv 3 - \tfrac{1}{2}\left[(3+4i)^{2n+1} + (3-4i)^{2n+1}\right] + 2^{4n+1} \pmod{100}.$$

Lemma 1. If α and β are integers, $\alpha \not\equiv 0 \pmod 5$ and if $\alpha^K \equiv \beta \pmod{100}$, then $\alpha^{K+20} \equiv \beta \pmod{100}$. However, if $\alpha = 2$, then K must be greater than 1.

Proof. Trivial.

In view of Lemma 1, for $n > 0$, we have

(9) $$2^{4n+1} \equiv 2^{4(n+5)+1} \pmod{100}.$$

Then we prove that

(10) $$\tfrac{1}{2}\{(3+4i)^{2n+1} + (3-4i)^{2n+1}\} \equiv \tfrac{1}{2}\{(3+4i)^{2n+11} + (3-4i)^{2n+11}\} \pmod{100}.$$

It is easy to see that the above congruence holds for modulus 4 hence we need to prove that

$$(3+4i)^{2n+1} + (3-4i)^{2n+1} \equiv (3+4i)^{2n+11} + (3-4i)^{2n+11} \pmod{25},$$

or

(11) $$(3+4i)^{2n+1}\{(3+4i)^{10} - 1\} + (3-4i)^{2n+1}\{(3-4i)^{10} - 1\} \equiv 0 \pmod{25}.$$

By actual expansion we find that

(12) $$(3+4i)^{10} - 1 \equiv 4(3-4i) \pmod{25}$$

and

$$(3-4i)^{10} - 1 \equiv 4(3+4i) \pmod{25}.$$

Let

(13) $$(3+4i)^{2n+1} = c + id, \quad (3-4i)^{2n+1} = c - id.$$

Expanding we find that

$$c = \sum_{r=0}^{n} \binom{2n+1}{2r} 3^{2n+1-2r}(-1)^r,$$

and

$$d = \sum_{r=0}^{n} \binom{2n+1}{2r+1} 3^{2n+1-(2r+1)} (-1)^r .$$

Lemma 2. $c \not\equiv 0 \pmod 5$ and $d \not\equiv 0 \pmod 5$.

Proof.

$$c \equiv \sum_{r=0}^{n} \binom{2n+1}{2r+1} (-2)^{2n+1-2r} (-1)^r \pmod 5$$

$$\equiv - \sum_{r=0}^{n} \binom{2n+1}{2r} 2^{2n+1-2r} (-1)^r \pmod 5$$

$$\equiv - \frac{(1-2i)^{2n+1} + (1+2i)^{2n+1}}{2i} \pmod 5 .$$

If $c \equiv 0 \pmod 5$ then since $5 = (1+2i)(1-2i)$ and hence $c \equiv 0 \pmod{1+2i}$, which is not true and hence $c \not\equiv 0 \pmod 5$. Similarly it can be proved that $d \not\equiv 0 \pmod 5$. Moreover from (13) we have

(14) $$c^2 + d^2 = (25)^{2n+1} \equiv 0 \pmod{25} .$$

Since $c \not\equiv 0$, $d \not\equiv 0 \pmod 5$ it is easy to see that $(c,d) = 1$ and hence there exist a number a such that

(15) $$c \equiv ad \pmod{25} .$$

Using (11) and (12), Eq. (10) simplifies to

(16) $$3c + 4d \equiv 0 \pmod{25} .$$

Therefore to prove (10), we need to prove (16). Substitute (15) into (14) to get $1 + a^2 \equiv 0 \pmod{25}$ which yields that either (a) $a \equiv 7 \pmod{25}$ or (b) $a \equiv 18 \pmod{25}$. We then prove that condition (a) can only be satisfied and thus will reject condition (b). Assume that (b) is satisfied, i.e., $c \equiv 18d \pmod{25}$ or

(17) $$c \equiv 3d \pmod 5 .$$

We show that (17) is impossible. We have proved that

$$c \equiv -\frac{(1-2i)^{2n+1} + (1+2i)^{2n+1}}{2i} \pmod{5} .$$

Similarly it can be proved that

$$D \equiv \frac{(1+2i)^{2n+1} + (1-2i)^{2n+1}}{2} \pmod{5} .$$

Substituting these expressions in (17) it can be easily proved that (17) will not even hold for modulus $(1 + 2i)$ or $(1 - 2i)$. Hence (17) is impossible and condition (b) cannot be satisfied. Therefore condition (a) has to be satisfied and hence $c \equiv 7d \pmod{25}$. Using this congruence we find that (16) is satisfied and hence we have proved the truth of (10). Using these results and (9), from (8) we get $S_{4n+2} \equiv S_{4n+22} \pmod{100}$. But $S_{4n+2} \equiv 52 \pmod{100}$ and therefore $S_{4n+22} \equiv 52 \pmod{100}$ and hence if (2) is true for $n > 0$ it is also true for $n + 5$. From Krick's [7] table for S_{2n} up to S_{20} we find that (2) is true for $n = 1, 2, 3, 4$. Also using (3) we verify that $S_{22} \equiv 52 \pmod{100}$. Thus by the usual method of induction (2) has been established.

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