THE CASE OF THE STRANGE BINOMIAL IDENTITIES OF PROFESSOR MORIARTY

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"My dear fellow," said Sherlock Holmes, as we sat on either side of the fire in his research library at Baker Street, "combinatorial identities are infinitely stranger than anything which the ordinary mortal mind can devise. If we could fly out of that window hand-in-hand, hover over some of the rare geniuses of mathematics, however, and peep in at the queer formulas boiling in their brains, the strange relations and inverse connections, vast chains of implications, we should see some very singular and ineluctable identities. Moreover, they form a beautiful order."

"And yet I am not convinced of it," I answered. "The formulas which appear in the literature are so numerous and diverse that I must quite agree with my old friend John Riordan [17, p. vii] who has often spoken of the protean nature of combinatorial identities. He has said that identities are both inexhaustible and unpredictable; and that the age-old dream of putting order in such a chaos is doomed to failure."

"A certain judicious selection must be made in order to exhibit the order which inheres in this subject," remarked Holmes. "This is wanting in research journals, where stress is placed on novelty and abstraction and the history of the subject is quite often laid entirely aside. A good detective of identities, however, remembers and retrieves numerous facts from the disarray of identities. This requires vast concentration and attention to detail. Depend upon it, behind every identity there is a whole history. As unofficial adviser to everyone who is puzzled by combinatorial identities, I come across many strange and bizarre formulas, none perhaps more strange than those formulas discovered by the infamous Professor Moriarty."

Holmes had now risen from his chair, and was standing before one of the enormous bookcases in his library, a library reputed to be filled with case histories of every binomial identity ever brought to trial. I could tell from his stance that he was about to embark on a story which would be both interesting and educational. He began speaking:

"Professor Moriarty first gave his formulas in the form of a dual pair:

$$\sum_{k=0}^{n-p} \binom{2n+1}{2p+2k+1}\binom{p+k}{k} = \binom{2n-p}{p} 2^{2n-2p} \tag{1}$$

and

$$\sum_{k=0}^{n-p} \binom{2n}{2p+2k}\binom{p+k}{k} = \frac{n}{2n-p}\binom{2n-p}{p} 2^{2n-2p} \tag{2}$$

I am indebted to E. T. Davis [7, p. 71] for calling them to my attention. Davis writes in a footnote that "We shall call these the identities of James Moriarty, since we do not know any other source from which such ingenious formulas could have come. See 'The Final Problem,' The Memoirs of Sherlock Holmes. "Here I have corrected formula (2) in that Davis' version would have $2p + k$ instead of the correct $2p + 2k$, something immediately obvious to an old combinatorial detective! I may say that I have also changed his summation variable from 's' to 'k' because my mind is stamped this way... it simplifies things."

Holmes paused and pulled out another book. He continued:

"The facts of Moriarty's life are well documented. Sabine Baring-Gould [3, pp. 21-23] has given them in a few lines. You see, Moriarty was my own teacher. As you know we developed into arch-enemies. Moriarty's mathematical work is summarized by Baring-Gould as follows:

> "At the age of twenty-one — in 1867 — this remarkable man had written a treatise on the binomial theorem which had a European vogue. On the strength of it — and because of certain connections his West of England family possessed — he won the mathematical chair at one of the smaller English universities. There he produced his magnum opus — a work for which, despite his later infamy, he will be forever famous. He became the author of 'The Dynamics of an Asteroid'."

It is clear from the further remarks given about this monumental work that a genius such as Moriarty could be responsible for formulas such as (1) and (2). Professor Davis, in a letter dated 29 July 1963, told me how elusive he found the proofs of (1) and (2) and could find no explicit reference in the literature. Actually there are many references, as will be clear in the list given below.

Moriarty was a master of disguise, and here we do find Riordan's remark about the protean nature of the identities pertinent. But it needs just a little care to see through the fabric. Replace k by $k - p$ and the two formulas become

$$\sum_{k=p}^{n}\binom{2n+1}{2k+1}\binom{k}{p} = \binom{2n-p}{p}2^{2n-2p} \tag{3}$$

and

$$\sum_{k=p}^{n}\binom{2n}{2k}\binom{k}{p} = \frac{n}{2n-p}\binom{2n-p}{p}2^{2n-2p}, \tag{4}$$

and in this form they appeal more to my experienced eye. As a matter of fact two slightly more general such formulas may be found in [12] where formulas (3.120) and (3.121) are as follows:

$$\sum_{k=j}^{[n/2]}\binom{n+1}{2k+1}\binom{k}{j} = 2^{n-2j}\binom{n-j}{j} \tag{5}$$

and

$$\sum_{k=j}^{[n/2]} \binom{n}{2k}\binom{k}{j} = 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j} \quad . \tag{6}$$

I first came upon (5) and (6) while studying elementary matrix theory. I was trying to determine the n^{th} power of a 2-by-2 matrix. If t_1 and t_2 denote distinct characteristic roots of such a matrix M, then it is easy to prove that

$$M^n = \frac{t_2^n - t_1^n}{t_2 - t_1} M - \frac{t_2^{n-1} - t_1^{n-1}}{t_2 - t_1} |M| I , \quad |M| = t_1 t_2 , \tag{7}$$

where $|M| = \det(M)$ and I is the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, by successively multiplying M times itself, one is led to conjecture and prove by induction that in fact

$$\begin{aligned} M^n = M &\sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k |M|^k \binom{n-k-1}{k} (t_1 + t_2)^{n-1-2i} \\ - |M| I &\sum_{k=0}^{\left[\frac{n-2}{2}\right]} (-1)^k |M|^k \binom{n-k-2}{k} (t_1 + t_2)^{n-2-2k} \quad . \end{aligned} \tag{8}$$

Upon equation (7) and (8) we obtain a single identity

$$\sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} (t_1 t_2)^j (t_1 + t_2)^{n-2j} = \frac{t_2^{n+1} - t_1^{n+1}}{t_2 - t_1} , \quad t_2 \neq t_1 \quad . \tag{9}$$

Separating out the even and odd index terms in the binomial expansion it is easy to obtain

$$\sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} z^k = \frac{(1 + \sqrt{z})^{n+1} - (1 - \sqrt{z})^{n+1}}{(1 + \sqrt{z}) - (1 - \sqrt{z})} ,$$

and we also have

$$\sum_{k=0}^{[n/2]} \binom{n+1}{2k+1}(x+1)^k = \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} \sum_{j=0}^{k} \binom{k}{j} x^j$$

$$= \sum_{j=0}^{[n/2]} x^j \sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j}.$$

More elegantly, we have proved that

$$\sum_{j=0}^{[n/2]} x^j \sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j} = \frac{t_2^{n+1} - t_1^{n+1}}{t_2 - t_1}, \tag{10}$$

where $t_2 = 1 + \sqrt{x+1}$, $t_1 = 1 - \sqrt{x+1}$. Since we have in this case $t_1 + t_2 = 2$, $t_1 t_2 = -x$, we may apply (10) to (9) when $t_1 + t_2 = 2$, and the result by equating coefficients of powers of x is the identity

$$\sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j} = 2^{n-2j}\binom{n-j}{j},$$

which is precisely formula (5) above. A natural companion piece may be found, and I use what I call an even and odd index argument to do this.

Indeed, write

$$S_n = \sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j}.$$

Now

$$\binom{n}{2k} + \binom{n}{2k+1} = \binom{n+1}{2k+1}$$

so that we have

$$\sum_{k=j}^{[n/2]} \binom{n}{2k}\binom{k}{j} + \sum_{k=j}^{[n/2]} \binom{n}{2k+1}\binom{k}{j} = \sum_{k=j}^{[n/2]} \binom{n+1}{2k+1}\binom{k}{j}.$$

Look at the second sum on the left: If n is odd $[n/2] = [(n-1)/2]$; but if n is even $[n/2] = [(n-1)/2] + 1$. It follows readily that we have proved

$$\sum_{k=j}^{[n/2]} \binom{n}{2k}\binom{k}{j} = S_n - S_{n-1} = 2^{n-2j}\binom{n-j}{j} - 2^{n-1-2j}\binom{n-1-j}{j}$$

$$= 2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j} \quad ,$$

in other words, we have proved formula (6) above.

The above proof was first obtained by me in the year 1950. It may or may not be original, as it is very difficult to guarantee originality. Formulas (5)-(6) arise naturally in the study of trigonometric identities. In the disguised forms (3) and (4) you will also find them in such studies. Glocksman and Ruderman [10] gave inductive proofs of (5) and (6). They have an interesting footnote calling attention to a pending Sherlock Holmes tale about these strange relations of Professor Moriarty, so that my present remarks are long overdue.

Trigonometric proofs are implicit in Bromwich [5, Chapter 9]. Such proofs are bound up with the well-known expansions

$$\cos nx = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} \frac{n}{n-j}\, 2^{n-2j-1} (\cos x)^{n-2j} \tag{11}$$

and

$$\frac{\sin(n+1)x}{\sin x} = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} 2^{2n-2j} (\cos x)^{n-2j} \quad . \tag{12}$$

See also Deaux [8] who works in reverse, using (6) to prove (11). Formula (1), with n replaced by n/2, was posed by André [1] and the solution given in 1871 by one Moret-Blanc suggests that this may be one of Moriarty's French disguises. Briones [4] was evidently unaware that the summation (6) could be done in closed form, as he carries the sum around consistently in unsummed form. Kaplansky, in a review of a paper of Gonzáles del Valle [11], restates the author's formula in our form somewhat like (1)... but note a few differences. This same related formula of Gonzáles del Valle occurs in [13]. Fred. Schuh [18] again gives something equivalent to (9) and obtains our formula (1) in only a slight variation. Singer [19] rediscovers a special case of the companion to (9) involving the coefficients

$$\frac{n}{n-j}\binom{n-j}{j} \quad ,$$

using formula (6), and cites Netto's famous Combinatorik [16].

The formulas in Netto [16, pp. 246-258] are taken from Father Eagen's valuable treatise [14, pp. 64-68]. It is curious to note that Netto attempts to correct some of Hagen's formulas and introduces further errors where errors sometimes did not appear. The basic error consists in failing to appreciate that $\binom{x}{n}$ is a polynomial of degree n in x, whence some of the inequalities appended by Netto are superfluous. Also, Netto omits one very striking formula (number 17) of Hagen, as being a linear combination of the others, which it is not. Hagen's formula (17) was the motivation of over 25 papers by me since 1956, that formula tracing to 1793 and one H. A. Rothe. A full discussion of the errors in Hagen and Netto must, however, await another time. At any rate, the formulas of Moriarty are in Netto and Hagen.

The connection of our formulas with Fibonacci-type polynomials and numbers should be clear because of formula (9) and the well-known formula

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} = F_{n+1} ,$$

where $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 0$, $F_1 = 1$, defining the Fibonacci numbers. As further evidence of this connection, Lind [15] has written to me of the following matter. Let $f_1(x) = 1$, $f_2(x) = x$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, denote the Fibonacci polynomials. It is known that

$$f_n(x) = \frac{1}{\sqrt{x^2+4}}\left\{\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n - \left(\frac{x-\sqrt{x^2+4}}{2}\right)^n\right\} .$$

What is more,

$$f_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

so that comparison of coefficients yields

$$\sum_{k=j}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1}\binom{k}{j} = 2^{n-2j-1}\binom{n-j-1}{j} ,$$

which, save for a shift of 1 in the index n, is precisely formula (5) above. Our detective work pays off. It shows that Moriarty is implicit in all the literature about Fibonacci polynomials and related generalizations.

As for further references in accessible literature, one should examine ex. 5, pp. 209-210 of Vol. 2 of Chrystal [6]. What is done there is to use an expansion on p. 202 for $t_2^n + t_1^n$

(instead of what we used in (9) above, thereby being a Lucas approach instead of a Fibonacci approach). Chrystal's formula may be written in the form

$$\sum_{s=0}\binom{m}{2r+2s}\binom{r+s}{s} = 2^{m-2r-1}\binom{m-r}{r}\frac{m}{m-r} ,$$

where I have set $n = m - 2r$ in his formulas. The result is, of course, our formula (6) in slight disguise. Both (5) and (6) are in Riordan [17, pp. 87, 243].

Moret-Blanc's solution to Andre's problem [1] finds something equivalent to our (5) as the coefficient of x^{2k+1} in the series expansion of

$$(1+x)^{n+1}\left(1-\frac{1}{x^2}\right)^{-k-1}$$

found in two ways.

Netto's proof of the formula in the form (same as in [11] and [13])

$$\sum_{s=0}\binom{p+s}{s}\binom{2p+m}{2p+2s+1} = 2^{m-1}\binom{m+p-1}{p} \tag{13}$$

is by equating coefficients in the algebraic identity

$$(1-x)^{-2p}\left\{\left(1-\frac{x}{1-x}\right)^2\right\}^{-p} = (1-2x)^{-p} . \tag{14}$$

Our formula (1) of Moriarty then follows by changing s into k and putting $m = 2n - 2p + 1$. A similar argument goes for the companion which he gives and which is of course equivalent to (2) here.

In our sleuthing of these old results we have come across one very old appearance in the literature of a Moriarty type formula in the form (6). The date of this case is 1826, a long time before the historic Moriarty appeared. Perhaps he lifted the results from such an earlier source, being an evil and crafty genius. In any event, Andreas von Ettingshausen, in his surprisingly modern book [9] gives the formula (page 257)

$$\mathfrak{S}_{0}^{r}\binom{n}{2r}\binom{r}{r-v} = \frac{n}{n-v}\binom{n-v}{v}2^{n-2v-1} . \tag{15}$$

This is the exact notation he uses. The German 'S' is used in place of Greek sigma for summation, and the indices of summation are arranged differently than modern form, but otherwise it is precisely the same formula. Ettingshausen gives many other formulas which are still today being rediscovered. Incidentally, Ettingshausen's book of 1826 has the first appearance in print of the modern symbol $\binom{n}{r}$ in place of the previously common symbols $\left(\frac{n}{r}\right)$ or $\left[\frac{n}{r}\right]$ used by Euler. Some historians have stated (Cajori notably) that the year was 1827 in another book by Ettingshausen, however the correct item appears to be the 1826 book.

We come now to a modern chapter: INVERSION. A good detective would not earn his pay if he did not make adroit use of inversion of series. Many such inverse series pairs exist, and we can do no better for the time than refer the reader to the excellent discussion by Riordan [17] for information on inverse series pairs of many types. We need one such, a very simple instance.

It is easy to prove by the use of the orthogonality relation

$$\sum_{k=r}^{j} (-1)^{k+j} \binom{k}{r}\binom{j}{k} = \begin{cases} 0, & j \neq r, \\ 1, & j = r, \end{cases}$$

that

$$\sum_{k=r}^{m} (-1)^{k} \binom{k}{r} f(k) = g(r) \tag{16}$$

if and only if

$$\sum_{k=r}^{m} (-1)^{k} \binom{k}{r} g(k) = f(r) .$$

The proof is nothing but inverting order of summation and using the stated orthogonality (cf. Riordan [17], p. 85). Taking

$$f(k) = (-1)^k \binom{n+1}{2k+1} \quad \text{and} \quad g(r) = 2^{n-2r}\binom{n-r}{r}, \quad m = \left[\frac{n}{2}\right],$$

we see that (5) inverts to yield

$$\sum_{k=r}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k}\binom{k}{r} 2^{n-2k} = (-1)^r \binom{n+1}{2r+1} . \tag{17}$$

Similarly, choosing

$$f(k) = (-1)^k \binom{n}{2k}, \quad g(r) = 2^{n-2r-1}\binom{n-r}{r}\frac{n}{n-r}, \quad m = \left[\frac{n}{2}\right],$$

we see that (6) inverts to yield

$$\sum_{k=r}^{\left[\frac{n}{2}\right]} (-1)^k \binom{k}{r} 2^{n-2k-1}\binom{n-k}{k}\frac{n}{n-k} = (-1)^r \binom{n}{2r}. \tag{18}$$

I have not noticed any significant appearance of these relations in the literature until now relation (17) has been found by Marcia Ascher [2] who has given an inductive proof. Naturally the skilled combinatorial detective expects a companion formula such as (18)."

Here Holmes rested and then concluded his story by saying that all of these relations are in turn special cases of much more beautiful ones, which must await another time and place. So much for the evil genius of Moriarty.

ADDENDUM

It may be of interest to show that two well-known Fibonacci formulas

$$F_{n+1} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \tag{19}$$

and

$$F_{n+1} = 2^{-n}\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2k+1} 5^k \tag{20}$$

may be derived, the one from the other, in either of two ways: by use of (5) or by use of (17). We also need the binomial theorem.

We have first, using (5) of the generalized Moriarty,

$$F_{n+1} = \sum_{k=0}^{[n/2]} \binom{n-k}{k} = \sum_{k=0}^{[n/2]} 2^{2k-n} \sum_{j=k}^{[n/2]} \binom{n+1}{2j+1}\binom{j}{k}, \quad \text{by (5)}$$

$$= 2^{-n} \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} \sum_{k=0}^{j} \binom{j}{k} 2^{2k} = 2^{-n} \sum_{j=0}^{[n/2]} \binom{n+1}{2j+1} 5^j ,$$

and the steps are reversible, showing (19) and (20) equivalent via (5).

We have next, using (17) of Ascher,

$$F_{n+1} = 2^{-n}\sum_{r=0}^{[n/2]}\binom{n+1}{2r+1}5^r = 2^{-n}\sum_{r=0}^{[n/2]}5^r\sum_{k=r}^{[n/2]}(-1)^{k+r}\binom{n-k}{k}\binom{k}{r}2^{n-2k}$$

$$= \sum_{k=0}^{[n/2]}\binom{n-k}{k}(-1)^k 2^{-2k}\sum_{r=0}^{k}\binom{k}{r}(-5)^r$$

$$= \sum_{k=0}^{[n/2]}\binom{n-k}{k}(-1)^k 2^{-2k}(-4)^k = \sum_{k=0}^{[n/2]}\binom{n-k}{k} \quad ,$$

and again the steps are reversible, showing (19) and (20) equivalent via (17). Similarly it is possible to use the Moriarty and inverse Moriarty relations to show other equivalences.

Thus relations (6) and (18) may be used precisely as above to show that the formulas

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n}{n-k}\binom{n-k}{k} = L_n \tag{21}$$

and

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}5^k = 2^{n-1}L_n \tag{22}$$

are equivalent. Here $L_{n+1} = L_n + L_{n-1}$, with $L_0 = 2$, $L_1 = 1$, define the Lucas numbers.

Another similar equivalence which follows from use of (5) or (17) is the pair of combinatorial identities

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-x}{n-k}\binom{n-k}{k}2^{n-2k} = \binom{2n-2x}{n} \tag{23}$$

and

$$\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n+1}{2j+1}\binom{n-x+j}{n} = \binom{2n-2x}{n} \tag{24}$$

valid for all real x. Identity (23) is essentially (3.107) in [12].

Again, the equivalence of the two summations

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n+k}{k}\binom{n-k}{k} 2^{n-2} = \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{j}\binom{n+1}{2j+1} \tag{25}$$

follows by use of (5) or by (17), although no closed form is known. By use of (5) it is easy to see that a closed form for

$$\sum_{k=0}^{[n/2]} \binom{n+k}{k}\binom{n-k}{k}$$

might depend on a closed form for

$$\sum_{k=0}^{j} \binom{n+k}{k}\binom{j}{k} 2^{2k} .$$

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