DISTRIBUTION OF FIBONACCI NUMBERS MOD 5^k

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It was shown by L. Kuipers and Jau-shyong Shiue [2] that the only moduli for which the Fibonacci sequence $\{F_n\}$, $n = 1, 2, \cdots$, can possibly be uniformly distributed are the powers of 5. In addition, the authors proved the Fibonacci sequence to be uniformly distributed mod 5, and they conjectured that this holds for all other powers of 5 as well. In this note, we settle this conjecture in the affirmative. Thus we show, in particular, that the Fibonacci sequence attains values from each residue class mod 5^k, and each residue class occurs with the same frequency. The weaker property of the existence of a complete residue system mod m in the Fibonacci sequence was investigated earlier by A. P. Shah [3] and G. Bruckner [1]. For definitions and terminology we refer to [2].

Theorem. The Fibonacci sequence $\{F_n\}$, $n = 1, 2, \cdots$, is uniformly distributed mod 5^k for all $k \geq 1$.

Before we start the proof, let us collect some useful preliminaries. It follows from a result of D. D. Wall [5, Theorem 5] that $\{F_n\}$, considered mod 5^k, has period $4 \cdot 5^k$. Therefore it will suffice to show that, among the first $4 \cdot 5^k$ elements of the sequence, we find exactly four elements, or, equivalently, at most four elements from each residue class mod 5^k. It will also be helpful to know that, for $j \geq 1$, the largest exponent e such that 5^e divides $(2j+1)!$ satisfies (see [4]):

$$e = \sum_{i=1}^{\infty} \left[\frac{2j+1}{5^i}\right] < \sum_{i=1}^{\infty} \frac{2j+1}{5^i} = \frac{2j+1}{4} < j \quad . \tag{1}$$

We note the formula

$$\binom{r+s}{t} = \sum_{i=0}^{t} \binom{r}{i}\binom{s}{t-i} \tag{2}$$

with non-negative integers r, s, and t, and $\binom{n}{j} = 0$ for $j > n$, which can be quickly verified by comparing the coefficients of x^t in $(1+x)^{r+s} = (1+x)^r(1+x)^s$.

Proof of the Theorem. We proceed by induction on k. For $k = 1$, the result was already shown in [2]. Now assume that, for some $k \geq 2$ and every integer a, the congruence $F_n \equiv a \pmod{5^{k-1}}$ has exactly four solutions c with $1 \leq c \leq 4 \cdot 5^{k-1}$. If n is a solution of $F_n \equiv a \pmod{5^k}$, $1 \leq n \leq 4 \cdot 5^k$, then $F_n \equiv a \pmod{5^{k-1}}$, hence by periodicity: $n \equiv c \pmod{4 \cdot 5^{k-1}}$ for one of the four c's. We complete the proof by showing that each value of c yields at most one solution n. For suppose we also have $F_m \equiv a \pmod{5^k}$,

$1 \le m \le 4 \cdot 5^k$, $m \equiv c \pmod{4 \cdot 5^{k-1}}$, and WLOG $n \ge m$. Then, in particular, $F_n \equiv F_m \pmod{5^k}$ and $n \equiv m \pmod{4 \cdot 5^{k-1}}$. Using the well-known representation

$$F_n = 2^{1-n} \sum_{j=0}^{\infty} 5^j \binom{n}{2j+1},$$

where $\binom{n}{r} = 0$ for $r > n$, we arrive at

$$\sum_{j=0}^{k-1} 5^j \binom{n}{2j+1} \equiv 2^{n-m} \sum_{j=0}^{k-1} 5^j \binom{m}{2j+1} \pmod{5^k} .$$

Since $2^{4 \cdot 5^{k-1}} \equiv 1 \pmod{5^k}$ by the Euler-Fermat Theorem, we get

$$\sum_{j=0}^{k-1} 5^j \left(\binom{n}{2j+1} - \binom{m}{2j+1} \right) \equiv 0 \pmod{5^k} . \tag{3}$$

We claim that, for $j \ge 1$, the corresponding term in this sum is divisible by 5^k. By (2):

$$5^j \left(\binom{n}{2j+1} - \binom{m}{2j+1} \right) = \sum_{i=1}^{2j+1} 5^j \binom{n-m}{i} \binom{m}{2j+1-i} .$$

We look at $5^j \binom{n-m}{i}$. From (1) we see that cancelling out 5's from $i!$ against 5^j leaves at least one power of 5 in the latter number. Since there is a factor 5^{k-1} in $n - m$, we get the desired divisibility property. Thus, from (3), we are left with the term corresponding to $j = 0$: $n - m \equiv 0 \pmod{5^k}$. Together with $n \equiv m \pmod{4 \cdot 5^{k-1}}$, this implies $n \equiv m \pmod{4 \cdot 5^k}$ or $n = m$.

ACKNOWLEDGEMENT

I wish to thank Prof. L. Kuipers for drawing my attention to this problem.

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