FIBONACCI NUMBERS OBTAINED FROM PASCAL'S TRIANGLE WITH GENERALIZATIONS

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1. INTRODUCTION

Consider the following array of numbers obtained from the first k lines of Pascal's Triangle.

$$\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 2 & 1 & \cdots & 0 \\
\hdashline
1 & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & 1 \\
\hline
k & \binom{k}{2} & \binom{k}{3} & \cdots & 1 \\
2k-1 & 2\binom{k}{2} & 2\binom{k}{3} & \cdots & 2 \\
4k-3 & 4\binom{k}{2}-1 & 4\binom{k}{3} & \cdots & 4 \\
\hdashline
\end{array}$$

If we let the element in the $i+1^{\text{th}}$ column and n^{th} row be $F_{i,n}$, then $F_{i,n} = \binom{n}{i}$ $(n, i = 0, 1, 2, \cdots, k-1)$ and

$$F_{i,n} = \sum_{j=1}^{k} F_{i,n-j} \qquad (i = 0, 1, 2, \cdots, k-1; \quad |n| = k, k+1, \cdots).$$

If $k = 2$, $F_{0,n} = f_{n+1}$, $F_{1,n} = f_n$, where f_n is the n^{th} Fibonacci number; and if $k = 3$, $F_{1,n} = L_{n+1}$ and $F_{2,n} = K_{n-1}$, where L_n and K_n are the general Fibonacci numbers of Waddill and Sacks [8]. Also, $F_{k-1,h} = f_{h,k}$ where the $f_{n,k}$ are the k-generalized Fibonacci numbers of Miles [5], and $F_{0,n} = U_{k,n}$ of Ferguson [2]. Both the numbers $f_{n,k}$ and $U_{k,n}$ are of use in polyphase merge sorting techniques (see, for example, Gilstad [3] and Reynolds [6]).

The purpose of this paper is to investigate some of the properties of a more general set of functions which include the functions $F_{i,n}$ $(i = 0, 1, 2, \cdots, k-1)$ and several others as special cases.

2. NOTATION AND DEFINITIONS

Let $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ be a fixed set of k integers such that

$$F(x) = x^k - \alpha_1 x^{k-1} + \alpha_2 x^{k-2} + \cdots + (-1)^k \alpha_k$$

has distinct zeros $\rho_0, \rho_1, \cdots, \rho_{k-1}$. Let $a_0, a_1, \cdots, a_{k-1}$ be any k integers and define

$$\phi_i = \sum_{j=0}^{k-1} a_j \rho_i^j \qquad (i = 0, 1, 2, \cdots, k-1) .$$

Finally, let

$$D = \begin{bmatrix} 1 & \rho_0 & \rho_0^2 & \cdots & \rho_0^{k-1} \\ 1 & \rho_1 & \rho_1^2 & \cdots & \rho_1^{k-1} \\ \hdashline 1 & \rho_{k-1} & \rho_{k-1}^2 & \cdots & \rho_{k-1}^{k-1} \end{bmatrix}$$

$$\Delta = |D| .$$

We shall concern ourselves with the functions

$$\text{(2.1)} \qquad A_{i,n} = \frac{1}{\Delta} \begin{vmatrix} 1 & \rho_0 & \cdots & \rho_0^{i-1} & \phi_0^n & \rho_0^{i+1} & \cdots & \rho_0^{k-1} \\ 1 & \rho_1 & \cdots & \rho_1^{k-1} & \phi_1^n & \rho_1^{i+1} & \cdots & \rho_1^{k-1} \\ \hdashline 1 & \rho_{k-1} & \cdots & \rho_{k-1}^{i-1} & \phi_{k-1}^n & \rho_{k-1}^{i+1} & \cdots & \rho_{k-1}^{k-1} \end{vmatrix}$$

$$(i = 0, 1, 2, \cdots, k-1) .$$

It is clear that

$$\text{(2.2)} \qquad A_{i,n} = \frac{1}{\Delta} \sum_{j=0}^{k-1} c_{ij} \phi_j^n \qquad (i = 0, 1, 2, \cdots, k-1) ,$$

where c_{ij} is the cofactor of ρ_i^j in D.

If $a_1 = 1$, $a_i = 0$ $(i = 0, 2, 3, \cdots, k-1)$, we have $\phi_i = \rho_i$; and, in this case, we define $A_{i,n}$ to be $z_{i,n}$. These functions, which are quite useful in the determination of the properties of $A_{i,n}$, have been dealt with in some detail by authors such as Bell [1], Ward [9] and Selmer [7]. When $\alpha_i = (-1)^{i+1}$ $(i = 1, 2, \cdots, k)$, $\{z_{k-1,n}\}$ is the general Fibonacci sequence discussed by Miles [5] and Williams [10].

Since matrix methods are advantageous in the treatment of the $A_{i,n}$ functions, we introduce the following:

$$C = \frac{1}{\Delta}\begin{bmatrix} c_{00} & c_{10} & \cdots & c_{k-1\ 0} \\ c_{01} & c_{11} & \cdots & c_{k-1\ 1} \\ \hdotsfor{4} \\ c_{0\ k-1} & c_{1\ k-1} & \cdots & c_{k-1\ k-1} \end{bmatrix},$$

$$C_i = \mathrm{diag}\,(c_{i0}, c_{i1}, \cdots, c_{i\ k-1}),$$

$$Z_i = \begin{bmatrix} z_{i,0} & z_{i,1} & \cdots & z_{i,k-1} \\ z_{i,1} & z_{i,2} & \cdots & z_{i,k} \\ \hdotsfor{4} \\ z_{i,k-1} & z_{i,k} & \cdots & z_{i,2k-2} \end{bmatrix}$$

$$P_{n,r} = \begin{bmatrix} \phi_0^n & \phi_1^n & \cdots & \phi_{k-1}^n \\ \phi_0^{n+r} & \phi_1^{n+r} & \cdots & \phi_{k-1}^{n+r} \\ \hdotsfor{4} \\ \phi_0^{n+(k-1)r} & \phi_1^{n+(k-1)r} & \cdots & \phi_{k-1}^{n+(k-1)r} \end{bmatrix}$$

$$B_{n,r} = \begin{bmatrix} A_{0,n} & A_{1,n} & \cdots & A_{k-1,n} \\ A_{0,n+r} & A_{1,n+r} & \cdots & A_{k-1,n+r} \\ \hdotsfor{4} \\ A_{0,n+(k-1)r} & A_{1,n+(k-1)r} & \cdots & A_{k-1,n+(k-1)r} \end{bmatrix}$$

$$B_{i,n,r} = \begin{bmatrix} A_{i,n} & & A_{i,n+(k-1)r} \\ A_{i,n+r} & A_{i,n+2r} & A_{i,n+kr} \\ \hdotsfor{3} \\ A_{i,n+(k-1)r} & A_{i,n+kr} & A_{i,n+(2k-2)r} \end{bmatrix}$$

3. SPECIAL CASES

The $A_{i,n}$ functions include a number of interesting functions as special cases. We have already mentioned the $z_{i,n}$ functions in the previous section and in this section we describe several other special cases. We first show the relation of the function $F_{i,n}$ to $A_{i,n}$.

Let H_j $(j = 1, 2, \ldots, k)$ be the j^{th} elementary symmetric function of $\phi_0, \phi_1, \cdots, \phi_{k-1}$, then

(3.1)
$$A_{i,n} = \sum_{j=1}^{k} (-1)^{j+1} H_j A_{i,n-j}$$

If

$$\alpha_i = (-1)^i \left[\binom{n}{i} - \binom{n}{i-1}\right] \qquad (i = 1, 2, \cdots, k)$$

and $a_0 = a_1 = 1$, $a_i = 0$ $(i = 2, 3, \cdots, k-1)$, we have $\phi_i = 1 + \rho_i$ and $H_j = (-1)^{j+1}$. Hence,

$$A_{i,n} = \sum_{j=1}^{k} A_{i,n-j}$$

and

$$A_{i,n} = \binom{n}{i} \qquad (0 \le i, \quad n \le k-1) ;$$

thus, in this case,

$$A_{i,n} = F_{i,n} .$$

When $k - 2$, $\alpha_1 = 1 - 2a$, $\alpha_2 = a^2 - a - 1$, $a_0 = a$, $a_1 = 1$, we have $H_1 = 1$, $H_2 = -1$, and $A_{0,n} = af_n + f_{n-1}$, $A_{i,n} = f_n$. If $\alpha_1 = 0$, $\alpha_2 = -5$, $a_0 = a_1 = 1$, we have

$$A_{0,n} = 2^{n-1}\ell_n \quad \text{and} \quad A_{1,n} = 2^{n-1}f_n ,$$

where ℓ_n is the n^{th} Lucas number. If (x_1, y_1) is the fundamental solution of the Pell equation

(3.2)
$$x^2 - dy^2 = 1 ,$$

$$\alpha_1 = 0, \quad \alpha_2 = -d, \quad a_0 = x_1, \quad a_1 = y_1 ,$$

Then $A_{0,n} = x_n$ and $A_{1,n} = y_n$, where (x_n, y_n) is the n^{th} solution of (3.2).

When $k = 3$, we also have some interesting cases. For example, if $\alpha_1 = \alpha_2 = \alpha_3 = 2$ and $a_0 = -a_1 = a_2 = 1$, we have $H_1 = -H_2 = H_3 = 1$ and $A_{0,n} = U_{3,n}$, $A_{1,n} = -L_{n-1}$, $A_{2,n} = f_{3,n+1} = K_n$. If (x_1, y_1, z_1) is a fundamental solution of the diophantine equation (Mathews [4])

(3.3)
$$x^3 + dy^3 + d^2z^3 - 3dxyz = 1 ,$$

and $\alpha_0 = \alpha_2 = 0$, $\alpha_3 = d$, $a_0 = x_1$, $a_1 = y_1$, $a_2 = z_1$, then all the solutions of (3.3) are given by

$$(A_{0,n}, A_{1,n}, A_{2,n}) \qquad (|n| = 0, 1, 2, \cdots) .$$

4. IDENTITIES

We now obtain several of the important relations satisfied by the $A_{i,n}$ functions. It will be seen that each of these relations is a generalization of a corresponding identity satisfied by the Fibonacci numbers. The most important properties of the Fibonacci numbers are the identities which connect the numbers f_{n+m}, f_{n-m} and f_{nm} to other Fibonacci numbers. For the sake of convenience, we shall call these relations the addition, subtraction, and multiplication formulas.

By (2.1),

$$B_{i,n+m,r} = P_{n,r}C_iP'_{m,r} ,$$

where we denote by B' the transpose of the matrix B. Since

$$CD = DC = I ,$$

$$B_{i,n+m,r} = P_{n,r}C'D'C_iDCP'_{m,r} = B_{n,r}Z_iB'_{m,r} ;$$

hence,

$$\begin{aligned} A_{i,m+n} &= (A_{0,n}A_{1,n}A_{2,n} \cdots A_{k,n})Z_i(A_{0,m}A_{1,m} \cdots A_{k,m})' \\ &= \sum_{h=0}^{k-1}\sum_{j=0}^{k-1} z_{i,h+j}A_{h,n}A_{j,m} \quad . \end{aligned} \tag{4.1}$$

This is the addition formula for $A_{i,n}$.

By the definition of $A_{i,n}$ it follows that

$$\phi_i^m = \sum_{j=0}^{k-1} A_{j,m}\rho_i^j ; \tag{4.2}$$

thus,

$$\begin{aligned} \rho_i^h\phi_i^m &= \sum_{j=0}^{k-1} \rho_i^{j+h} A_{j,m} \\ &= \sum_{j=0}^{k-1} \rho_i^h \left(\sum_{j=0}^{k-1} z_{h,j+k} A_{j,m}\right) \quad . \end{aligned}$$

Now if $H = H_k = \phi_0\phi_1\phi_1 \cdots \phi_{k-1}$,

$$H^m A_{i,n-m} = \frac{1}{\Delta}\begin{vmatrix} \phi_0^m & \rho_0\phi_0^m & \cdots & \rho_0^{i-1}\phi_0^m & \phi_0^n & \rho_0^{i+1}\phi_0^m & \cdots & \rho_0^{k-1}\phi_0^m \\ \phi_1^m & \rho_1\phi_1^m & \cdots & \rho_1^{i-1}\phi_1^m & \phi_1^n & \rho_1^{i-1}\phi_1^m & \cdots & \rho_1^{k-1}\phi_1^m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{k-1}^m & \rho_{k-1}\phi_{k-1}^m & \cdots & \rho_{k-1}^{i-1}\phi_{k-1}^m & \phi_{k-1}^m & \rho_{k-1}^{i+1}\phi_{k-1}^m & \cdots & \rho_{k-1}^{k-1}\phi_{k-1}^m \end{vmatrix}$$

By (4.2)

(4.3) $H^m A_{i,n-m} =$

$$\begin{vmatrix} \sum_{j=0}^{k-1} z_{0,j}A_{j,m} & \cdots & \sum z_{0,i+j-1}A_{j,m} & A_{0,m} & \sum z_{0,i+j+1}A_{j,m} & \cdots & \sum z_{0,i+k-1}A_{j,m} \\ \sum z_{i,j}A_{j,m} & \cdots & \sum z_{1,i+j-1}A_{j,m} & A_{1,m} & \sum z_{1,i+j+1}A_{j,m} & \cdots & \sum z_{1,i+k-1}A_{j,m} \\ \sum z_{k-1,j}A_{j,m} & \cdots & \sum z_{k-1,i+j-1}A_{j,m} & A_{k-1,m} & \sum z_{k-1,i+j+1}A_{j,m} & \cdots & \sum z_{k-1,i+k-1}A_{j,m} \end{vmatrix}$$

this is the subtraction formula for $A_{i,n}$.

Since

$$A_{i,nm} = \frac{1}{\Delta}\sum_{j=0}^{k-1} c_{ij}\phi_j^{nm} ,$$

$$A_{i,nm} = \sum_{j=0}^{k} (\Delta^{-1}c_{ij})\left(\sum_{h=0}^{k-1} A_{h,m}\rho_j^h\right)^m .$$

From (4.2), we get the multiplication formula

$$A_{i,nm} = \sum \frac{m!}{i_1!i_2! \cdots i_k!}\prod_{j=1}^{k} A_{j,m}^{i_j} z_{i,s} ,$$

where the sum is taken over all non-negative integers $i_1, i_2, \cdots, i_k$ such that $\sum i_k = m$; and $\sum (j-1)i_j$.

We may easily evaluate the determinants of $B_{n,r}$ and $B_{i,n,r}$ by first introducing the matrix.

$$Q_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \phi_0^n & \phi_1^n & \cdots & \phi_{k-1}^n \\ \cdots & \cdots & \cdots & \cdots \\ \phi_0^{(k-1)n} & \phi_1^{(k-1)n} & \cdots & \phi_{k-1}^{(k-1)n} \end{vmatrix}$$

Now

$$Q_n C = \begin{vmatrix} A_{0,0} & A_{1,0} & \cdots & A_{k-1,0} \\ A_{0,n} & A_{1,n} & \cdots & A_{k-1,n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{0,(k-1)n} & A_{1,(k-1)n} & \cdots & A_{k-1,(k-1)n} \end{vmatrix}$$

hence

$$|Q_n| = \Delta \begin{vmatrix} A_{1,n} & \cdots & A_{k,n} \\ A_{1,2n} & \cdots & A_{k,2n} \\ \cdots & \cdots & \cdots \\ A_{1,(k-1)n} & \cdots & A_{k,(k-1)n} \end{vmatrix}$$

Since

$$|P_{n,r}| = H^n |Q_r| ,$$

we have

$$|B_{n,r}| = \frac{H^n}{\Delta} \begin{vmatrix} A_{1,r} & \cdots & A_{n-1,r} \\ A_{1,2r} & \cdots & A_{n-1,2r} \\ \cdots & \cdots & \cdots \\ A_{1,(k-1)r} & \cdots & A_{n-1,(k-1)r} \end{vmatrix} \tag{4.5}$$

and

$$|B_{i,n,r}| = H^n |Z_i| \begin{vmatrix} A_{1,r} & \cdots & A_{n-1,r} \\ A_{1,2r} & \cdots & A_{n-1,2r} \\ \cdots & \cdots & \cdots \\ A_{1,(k-1)r} & \cdots & A_{n-1,(k-1)r} \end{vmatrix}^2 . \tag{4.6}$$

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...ANNOUNCEMENT...

The Editor (who always parks in a Fibonacci-numbered parking space) noted the following in the latest publication of the Fibonacci Association, A Primer for the Fibonacci Numbers: There are 13 authors, each of whom wrote a Fibonacci number of articles. Each co-author has a Fibonacci number of articles with a given co-author. There are 11 articles with one author, and 13 articles have co-authors. Of the twenty-four articles, 13 are Primer articles, and 11 are not.

The Primer, co-edited by Marjorie Bicknell and V. E. Hoggatt, Jr., is a compilation of elementary articles which have appeared over the years. These articles were selected and edited to give the reader a comprehensive introduction to the study of Fibonacci sequences and related topics. The 175-page Primer will be available in the Fall of 1972 at a cost of $5.00.