A GENERALIZED FIBONACCI NUMERATION

E. ZECKENDORF
Liège, Belgium

The sequence of Fibonacci numbers is defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$ $(n \geq 0)$, and it is well known that

$$(1) \qquad F_n = \sum_{s=0}^{m} \binom{n-1-s}{s}, \quad \text{where } m \text{ is the greatest integer } \leq (n-1)/2\,.$$

It can be shown, if we allow negative values of the subscript n that

$$(2) \qquad F_{-n} = (-1)^{n-1} \cdot F_n\,.$$

Any sequence satisfying the recurrence relation

$$(3) \qquad t_{n+2} = t_n + t_{n+1}$$

is called a "generalized Fibonacci sequence." As soon as the values of any two consecutive terms $t_n = p$ and $t_{n+1} = q$ have been chosen, one can prove by induction that

$$(4) \qquad t_{n+s} = p \cdot F_{s-1} + q \cdot F_s \quad \text{and} \quad t_{n-s} = (p \cdot F_{s+1} - q \cdot F_s) \cdot (-1)^s\,.$$

Note that the subscript n is assumed to run from $-\infty$ to $+\infty$ in the generalized Fibonacci sequences as well as in the sequence of Fibonacci numbers.

a. Whenever t_n and t_{n+1} have the same sign, all terms t_{n+s} $(s \geq 2)$ are positive or negative and their absolute value increases with s.

Let us take, for example, the sequence $t_n = \alpha^n$, where $n \in (-\infty, +\infty)$ and α is a positive real number >1: every term is positive and they increase to infinity.

(Since $t_0 = 1$, $t_1 = \alpha$ and $t_2 = \alpha^2$, α must be the positive root of the "quadratic Fibonacci equation" $\alpha^2 - \alpha - 1 = 0$, that is,

$$\alpha = \frac{1 + \sqrt{5}}{2}\,.$$

Note that the sequence $t_n = \beta^n$, where

$$\beta = \frac{1 - \sqrt{5}}{2}$$

is the negative root, also satisfies the recurrence (3).)

b. If one of the terms t_n and t_{n+1} is positive and the other negative, and if we assume that $|t_n| > |t_{n+1}|$, then $|t_{n+2}| = |t_n| - |t_{n+1}|$ and the following terms **have alternated** signs.

Let us take, for example, the sequence $t_n = \beta^n$, where $n \in (-\infty, +\infty)$ and β is negative and smaller than 1: the terms of this sequence have alternated signs and their absolute value tends to zero when n goes to infinity.

c. In a generalized Fibonacci sequence where positive and negative terms alternate, if there is a term t_{n+1} such that $|t_{n+1}| > |t_n|$, both t_{n+1} and t_{n+2} will have the same sign. So t_{n+1} will start an infinite sequence all of whose terms are either positive or negative, with their absolute values increasing to infinity.

d. In a sequence with alternating positive and negative terms, if there is a term t_{n+1} such that $|t_{n+1}| = |t_n|$, the next term is 0; the terms of this sequence are clearly multiples of the Fibonacci numbers.

Except the sequencex α^n and β^n defined above, the generalized Fibonacci sequences have those two infinite parts: the lower part with alternating terms decreasing in absolute value, followed by the upper part whose terms have the same sign and increase in absolute value.

Let ϵ denote any positive or negative real number. It can be shown that a sequence where $t_0 = \alpha^0$ and $t_1 = \alpha^1 + \epsilon$ has to start with alternating positive and negative terms: for ϵ arbitrarily small, and for n large enough, in

$$t_{-n} = \alpha^{-n} + (-1)^{n-1} \cdot \epsilon \cdot F_n, \qquad \left|\epsilon \cdot F_n\right| > \alpha^{-n}.$$

On the other hand, a sequence where $t_0 = \beta^0$ and $t_1 = \beta^1 + \epsilon$ ends with terms increasing in absolute value, all of them being either positive or negative: for ϵ arbitrarily small and for n large enough, in

$$t_n = \beta^n + \epsilon \cdot F_n, \qquad \left|\epsilon \cdot F_n\right| > \left|\beta^n\right|.$$

We shall call "primary generalized Fibonacci sequences" those sequences which have at least one term equal to 1. It is no loss of generality to assume $t_0 = 1$. (Among Fibonacci numbers, three (F_{-1}, F_1 and F_2) are equal to 1: any may be the chosen t_0.) For these sequences we may write $t_0 = 1$, $t_1 = q$ and (4) becomes

$$t_s = F_{s-1} + q \cdot F_s \quad \text{and} \quad t_{-s} = (F_{s+1} - q \cdot F_s) \cdot (-1)^s. \tag{5}$$

In this paper, we intend to express the natural numbers 1, 2, 3, $\cdots$ as sums of distinct non-consecutive terms of primary generalized Fibonacci sequences and we shall obtain a coherent system of numeration that could be used in arithmetical operations.

We assumed $t_0 = 1$; all other terms are still undetermined; they might all be positive or negative or they might alternate in sign, and their values might increase or decrease; the recurrent relation (3) will be the only rule. Thus their expression is of an algebraic nature; the value of t_0 only has been fixed. Any other given term may take different values and in general, it is not possible to determine the sum of several given terms, when the sequence they are taken from is unknown. We shall see that the groups of terms belonging to the Generalized Fibonacci Numeration constitute an exception.

The natural numbers will be constructed by successive additions of the unit t_0. More precisely, two rules will be used, one for consecutive terms, namely $t_x + t_{x+1} = t_{x+2}$ and the other for equal terms, namely $2t_x = t_x + (t_{x-1} + t_{x-2}) = t_{x+1} + t_{x-2}$.

Two different notations are possible for a number N.

In the first one, let t_x be the term with the highest subscript and t_{-s} that with lowest subscript used in the expression of N. For each of these terms and for all terms between t_x and t_{-s} (taken from left to right), let us use the digit 1 for the terms involved in the expression of N and the digit 0 for every other term. For convenience of reading, we shall distinguish by punctuation the digit corresponding to t_0 and arrange the other digits in groups of four.

In the second one, the terms involved in the expression of N only are listed as subscripts of the letter t.

For instance, both 10.0100.0.1001.01 and $t_{6,3,-1,-4,-6}$ will represent the number N = 23.

Later on, when arithmetical operations are performed, it may be convenient to avoid the letter t, the expression of N being then shortened to the sequence of subscripts of t.

Applying these rules, we write successively

$$
\begin{aligned}
1 &= t_0 \\
2 &= t_0 + t_0 = t_{1,-2} \\
3 &= t_{1,-2} + t_0 = t_{2,-2} \\
4 &= t_{2,-2} + t_0 = t_{2,0,-2} \\
5 &= t_{2,0,-2} + t_0 = t_2 + (t_1 + t_{-2}) + t_{-2} = t_{3,-1,-4} \\
6 &= t_{3,-1,-4} + t_0 = t_{3,1,-4} \\
7 &= t_{3,1,-4} + t_0 = t_{4,-4} \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

These numbers are the groups of terms belonging to the Generalized Fibonacci Numeration (G. F. N.): they represent always the same number and they do not depend on the primary generalized Fibonacci sequence which has been chosen. The other groups of terms do not have this property. One should be able to recognize those particular groups of terms, and so we shall describe them.

We have first to explain how the joined Fibonacci terms taken from various undetermined sequences can yield the same value when added up in a formula of the G. F. N.

The above formulas were constructed by successive additions of the unit t_0 and, according to (4) and (5),

$$t_0 = F_{-1} + q \cdot F_0 = 1 \quad . \tag{6}$$

Let us look for the part played by $t_0 = F_{-1} = 1$ and $t_1 = q \cdot F_1 = q$, respectively in the construction of the numbers in the G. F. N.

Formula (6) consists of two parts: the first part, $F_{-1} = 1$, generates the formula for N when the terms t_n are given the value F_{n-1}. The part played by $t_1 = q$ is 0; it is represented in t_0 by $q \cdot F_0$ and in N by a similar expression, when the terms t_n are given the value F_n.

Example a. For $t_n = F_{n-1}$, $t_{6,3,-1,-4,-6} = 23$:

$$\begin{aligned} t_{6,3,-1,-4,-6} &= T_5 + F_2 + F_{-2} + F_{-5} + F_{-7} \\ &= F_7 + 2F_5 \quad \text{according to (2) and} \\ &= F_8 + F_3 = 23 \ . \end{aligned}$$

Example b. For $t_n = F_n$, $t_{6,3,-1,-4,-6} = 0$:

$$\begin{aligned} t_{6,3,-1,-4,-6} &= F_6 + F_3 + F_{-1} + F_{-4} + F_{-6} \\ &= F_3 + F_1 - F_4 \quad \text{according to (2), then by adding } F_0 = 0: \\ &= F_3 + F_1 + F_0 - F_4 = F_4 - F_4 = 0 \ . \end{aligned}$$

We shall now describe the numbers in the G. F. N. The reader is advised to construct the table of the first 50 natural numbers represented by the subscripts of t. (This table can also be found hereafter.) To be clearer, all terms with the same subscript will be written in the same column of the table when they are involved in the expression of several numbers.

Description of the numbers in the G. F. N. Every number is built from one or more independent groups of terms. First we have to describe these groups.

A. The symmetric groups contain

a) the term $t_0 = 1$,

b) the symmetric pairs with even subscripts (e.g., $t_{2,-2}$, $t_{4,-4}$, $\cdots$). These pairs stand for the numbers 3, 7, 18, $\cdots$, that is the Lucas numbers L_{2n}. One or more symmetric group (e.g., $t_{6,0,-6}$, $t_{3,4,-4,-8}$).

When some symmetric pairs and the term t_0 get together without gap, they form the saturated symmetric groups (e.g., $t_{6,4,2,0,-2,-4,-6}$). These saturated groups stand for the numbers 4, 11, 29, $\cdots$, that is the numbers L_{2n+1}.

(Actually, the sum of a symmetric pair and of the corresponding saturated symmetric group gives rise to the next symmetric pair: $t_{6,-6} + t_{6,4,2,0,-2,-4,-6} = t_{8,-8}$.)

B. The asymmetric groups are distinguished by their extreme terms, the upper one with a positive odd subscript $u(t_u)$ and the lower one with negative and even subscript $u+1$ $(t_{-(u+1)})$. The intermediate terms characterize the varieties of asymmetric groups:

a) In the typical asymmetric group, all intermediate terms have negative odd subscripts following one another without gap from t_{-1} to $t_{-(u-2)}$. These numbers arise by adding the unit t_0 to a saturated symmetric group:

$$t_0 + t_0 = t_{1,-2}\,; \qquad t_{4,2,0,-2,-4} + t_0 = t_{5,-1,-3,-6}\,.$$

b) In the usual asymmetric group, one or more intermediate terms have a positive subscript. These terms replace the symmetric terms with negative odd subscript. By adding the saturated symmetric group $t_{2n,2n-2,\cdots,-2n}$, the intermediate term t_{2n+1} replaces the term with negative symmetric subscript:

$$t_{9,-1,-3,-5,-7,-10} + t_{4,2,0,-2,-4} = t_{9,5,-1,-3,-7,-10}\,.$$

So it is possible to get the asymmetric saturated group (e.g., $t_{9,7,5,3,1,-10}$) where all the intermediate terms have undergone this substitution.

The intermediate terms of the usual asymmetric group are the next ones: $t_{\pm 1}$, $t_{\pm 3}, \cdots, t_{\pm(u-2)}$.

c) In the altered asymmetric group, the presence of an intermediate term with positive even subscript coincides with the suppression of the terms of odd and lower subscript (in absolute value). To change an asymmetric group to such an extent, one has to add the asymmetric saturated group, immediately prior to the new term with even subscript:

$$t_{11,9,-1,-3,-5,-7,-12} + t_{5,3,1,-6} = t_{11,9,6,-7,-12}\,.$$

C. The associated groups. We have seen that symmetric pairs can join with or without t_0 in order to form symmetric groups. Symmetric pairs may also surround the terms of an asymmetric group:

$$t_{6,-6} + t_{3,1,-4} = t_{5,3,1,-4,-6}\,.$$

Usually, nothing of this type occurs with asymmetric groups: the presence of the intermediate terms prevents it. Yet in the altered asymmetric group, the interval between the new term with even subscript and the terms with negative subscripts left over, this interval may be adequate for another group of terms:

$$t_{9,6,-7,-10} + t_{4,0,-4} = t_{9,6,4,0,-4,-7,-10}\,.$$

Estimation of the numbers in the G. F. N. In presence of joined generalized Fibonacci terms, when we have identified a number of the G. F. N., we have to find out its precise value.

A primary generalized Fibonacci sequence may be chosen so as to assign a fixed value to each term as we did with Fibonacci numbers. However, in the next step we had to add or subtract the terms with negative subscript, according to their parity. The difficulty in restoring the formula is still more significant.

Let us try to estimate these formulas by another method involving the terms with nonnegative subscript. The precise estimation of a number has to be made by a reckoning process, in spite of the undetermined value of the components.

a) We did assign to t_0 the value 1 and to the symmetric pairs the values 3, 7, 18, $\cdots$, which are those of L_2, L_4, L_6, $\cdots$, in the sequence of Lucas numbers. Let us relate these values to the terms with positive subscript t_2, t_4, t_6, $\cdots$ in these pairs. Then we may disregard the terms with negative subscript and all the symmetric groups will be correctly estimated.

b) Is it likewise possible to estimate the asymmetric groups?

1. Let $t_1 = 1$, $t_3 = 4$, $t_5 = 11$, $\cdots$ in other words, the Lucas numbers L_1, L_3, L_5, $\cdots$; these values were already assigned to the symmetric saturated groups; they are an underestimation for the typical asymmetric groups obtained by adding the unit t_0 to the symmetric saturated groups.

2. An intermediate term with positive subscript is substituted to the one with negative subscript by adding a symmetric saturated — and thus correctly estimated — group. Hence the underestimation of the asymmetric group persists.

3. When an underestimated asymmetric group is altered by adding an asymmetric saturated group that is likewise underestimated, an intermediate term with positive and even subscript appears: this term makes up for the two underestimations of one unit t_0.

As a matter of fact, the values L_1, L_3, $\cdots$, L_{2n-1} have been assigned to the terms with positive subscript t_1, t_3, $\cdots$, t_{2n-1} and the value L_{2n} to the new term with even subscript, t_{2n}. Now, $L_{2n} - (L_1 + L_3 + \cdots + L_{2n-1}) = L_0 = 2$.

Therefore, the altered asymmetric group is correctly estimated and the foreseen estimation of the number N is possible.

i. To the existing terms with positive subscript, we assign the next values: to t_0 the value 1; to the next terms t_1, t_2, t_3, $\cdots$, respectively, the values 1, 3, 4, 7, $\cdots$ of the Lucas numbers.

ii. It remains to add one unit to the sum of these estimations when the number N contains an unaltered asymmetric group. A single one only can exist in the expression for the number N. Therefore, this unit depends on the term with lowest positive subscript: when this subscript is odd, the unit has to be added.

Expression of a natural number by the G. F. N. We now possess all required elements to find such an expression. Let us consider, for instance, the numbers 59 and 87.

$59 = t_8 + t_5 + 1$ $\quad$ t_5 being the term with lowest positive subscript, the unit is needed to correct the underestimation.

t_8 belongs to the symmetric pair $t_{8,-8}$; t_5 belongs to the typical symmetric group $t_{5,-1,-3,-6}$. Hence the formula: $t_{8,5,-1,-3,-6,-8}$.

$87 = t_9 + 11$	We cannot use t_5: 5 being odd, one unit more would be needed to correct the underestimation.
$87 = t_9 + t_4 + 4$	For the same reason, we cannot use t_3 and the use of adjoining terms is not allowed.
$87 = t_9 + t_4 + t_2 + 1$	The last unit has to be t_0.
$87 = t_9 + t_4 + t_2 + t_0$.	

t_9 and t_4 belong to the altered asymmetric group $t_{9,4,-5,-7,-10}$: t_2 and t_0 belong to the saturated symmetric group $t_{2,0,-2}$. Hence the formula: $t_{9,4,2,0,-2,-5,-7,-10}$.

Remark. The recurrence relation (3) prevented us from using adjoining terms. An investigation of the numbers in the G. F. N. will show one more peculiarity of this numeration: t_0 does never follow directly a term with odd subscript.

THE FIRST 50 GENERALIZED FIBONACCI NUMBERS. List of Subscripts.

1:	0	21:	6 . 2 . . -2 . -6
2:	1. . -2	22:	6 . 2 .0. -2 . -6
3:	2 . . -2	23:	6 . 3 . .-1 -4. -6
4:	2 .0. -2	24:	6 . 3 1. . -4. -6
5:	3 . .-1 -4	25:	6 . 4 . . -4. -6
6:	3 1. . -4	26:	6 . 4 .0. -4. -6
7:	4 . . -4	27:	6 . 4 1. . -2 -4. -6
8:	4 .0. -4	28:	6 . 4 2 . . -2 -4. -6
9:	4 1. . -2 -4	29:	6 . 4 2 .0. -2 -4. -6
10:	4 2 . . -2 -4	30:	7 . . .-1 -3 .-5 -8
11:	4 2 .0. -2 -4	31:	7 . 1. . -3 .-5 -8
12:	5. . .-1 -3 . -6	32:	7 . 2 . . -3 .-5 -8
13:	5. 1. . -3 . -6	33:	7 . 2 .0. -3 .-5 -8
14:	5. 2 . . -3 . -6	34:	7 . 3 . .-1 .-5 -8
15:	5. 2 .0. -3 . -6	35:	7 . 3 1. . .-5 -8
16:	5. 3 . .-1 . -6	36:	7 . 4 . . .-5 -8
17:	5. 3 1. . . -6	37:	7 . 4 .0. .-5 -8
18:	6 . . . . -6	38:	7 . 4 1. . -2 .-5 -8
19:	6 . .0. . -6	39:	7 . 4 2 . . -2 .-5 -8
20:	6 . 1. . -2 . -6	40:	7 . 4 2 .0. -2 .-5 -8

(Continued on the following page.)

41:	7	5.			.	.	-1	-3	.	-8	46:	7	5.	3		1.	.		.	-8
42:	7	5.			1.	.		-3	.	-8	47:	8	.			.	.		.	-8
43:	7	5.		2	.	.		-3	.	-8	48:	8	.			.	0.		.	-8
44:	7	5.		2	.	0.		-3	.	-8	49:	8	.			1.	.	-2	.	-8
45:	7	5.	3		.	.	-1		.	-8	50:	8	.		2	.	.	-2	.	-8

[Continued from page 364.]

<u>Example:</u> $F_5 = 5$ and $2 \cdot 3 \cdot 7 \cdot 18 \cdot 47 = 35{,}532 \equiv 2 \pmod 5$.

The congruence is reminiscent of the congruences of Wilson and Fermat.

It is expected that many other interesting and novel consequences follow from the extended Hermite theorems (6.2) and (7.1) giving arithmetic information about Fibonacci, Lucas and other similar numbers.

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